Foundational, Compositional (Co)datatypes for Higher-Order Logic
Category Theory Applied to Theorem Proving

Dmitriy Traytel      Andrei Popescu      Jasmin Blanchette
Outline

Datatypes in HOL—State of the Art

Bounded Natural Functors

(Co)datatypes

(Co)nclusion
Outline

Datatypes in HOL—State of the Art

Bounded Natural Functors

(Co)datatypes

(Co)nclusion
Isabelle/HOL

- LCF philosophy
Isabelle/HOL

- LCF philosophy
  
  *Small inference kernel*
Isabelle/HOL

- LCF philosophy
  *Small inference kernel*
- Foundational approach
Isabelle/HOL

- LCF philosophy
  
  *Small inference kernel*

- Foundational approach
  
  *Reduce high-level specifications to primitive mechanisms*
Isabelle/HOL

- LCF philosophy
  *Small inference kernel*
- Foundational approach
  *Reduce high-level specifications to primitive mechanisms*
- HOL = simply typed set theory with ML-style polymorphism
Isabelle/HOL

- LCF philosophy
  *Small inference kernel*
- Foundational approach
  *Reduce high-level specifications to primitive mechanisms*
- HOL = simply typed set theory with ML-style polymorphism
  *Restrictive logic*
Isabelle/HOL

- LCF philosophy
  *Small inference kernel*

- Foundational approach
  *Reduce high-level specifications to primitive mechanisms*

- HOL = simply typed set theory with ML-style polymorphism
  *Restrictive logic*
  *Weaker than ZF*
Isabelle/HOL

- LCF philosophy
  *Small inference kernel*

- Foundational approach
  *Reduce high-level specifications to primitive mechanisms*

- HOL = simply typed set theory with ML-style polymorphism
  *Restrictive logic*
  *Weaker than ZF*
• Datatype specification

  datatype $\alpha$ list = Nil | Cons $\alpha$ ($\alpha$ list)

  datatype $\alpha$ tree = Node $\alpha$ ($\alpha$ tree list)
• Datatype specification

\[
\begin{align*}
\text{datatype } \alpha \text{ list} & = \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list}) \\
\text{datatype } \alpha \text{ tree} & = \text{Node } \alpha (\alpha \text{ tree list})
\end{align*}
\]

• Primitive type definitions
The traditional approach

Melham 1989, Gunter 1994

- Fragment of ML (non-co)datatypes
The traditional approach

Melham 1989, Gunter 1994

- Fragment of ML (non-co)datatypes
- Fixed universe for recursive types
The traditional approach

Melham 1989, Gunter 1994

- Fragment of ML (non-co)datatypes
- Fixed universe for recursive types
- Simulate nested recursion by mutual recursion

\[
\begin{align*}
\text{datatype } \alpha \text{ list } & = \text{Nil | Cons } \alpha (\alpha \text{ list}) \\
\text{datatype } \alpha \text{ tree } & = \text{Node } \alpha (\alpha \text{ tree list})
\end{align*}
\]
The traditional approach
Melham 1989, Gunter 1994

- Fragment of ML (non-co)datatypes
- Fixed universe for recursive types
- Simulate nested recursion by mutual recursion

\[
\begin{align*}
\text{datatype } \alpha \text{ list} &= \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list}) \\
\text{datatype } \alpha \text{ tree} &= \text{Node } \alpha (\alpha \text{ tree_list}) \\
\text{and } \alpha \text{ tree_list} &= \text{Nil} \mid \text{Cons } (\alpha \text{ tree}) (\alpha \text{ tree_list})
\end{align*}
\]
The traditional approach
Melham 1989, Gunter 1994

- Fragment of ML (non-co)datatypes
- Fixed universe for recursive types
- Simulate nested recursion by mutual recursion

```
datatype α list = Nil | Cons α (α list)
datatype α tree = Node α (α tree_list)
and α tree_list = Nil | Cons (α tree) (α tree_list)
```

- Implemented in Isabelle by Berghofer & Wenzel 1999
Limitations

Berghofer & Wenzel 1999

1. noncompositionality
2. no codatatypes
3. no non-free structures
Limitations

LICS 2012

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Limitations

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Bounded Natural Functors

(Co)datatypes

(Co)clusion
Datatypes in HOL—State of the Art

Bounded Natural Functors

(Co)datatypes

(Co)clusion

datatype $\alpha$ list  $=$  Nil $|$ Cons $\alpha$ ($\alpha$ list)

codatatype $\alpha$ tree  $=$  Node $\alpha$ ($\alpha$ tree list)
datatype \( \alpha \) list = Nil | Cons \( \alpha \) (\( \alpha \) list)

codatatype \( \alpha \) tree = Node \( \alpha \) (\( \alpha \) tree list)

- \( P \ n = \text{print} \ n; \text{for } i = 1 \text{ to } n \text{ do } P \ (n + i); \)
datatype $\alpha$ list $= \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list})$

codatatype $\alpha$ tree $= \text{Node } \alpha (\alpha \text{ tree list})$

- $P \ n = \text{print } n; \text{ for } i = 1 \text{ to } n \text{ do } P (n + i)$;
- evaluation tree for $P \ 2$

![Diagram]

Compositionality = no unfolding

Need abstract interface

What interface?
datatype \( \alpha \) list = Nil | Cons \( \alpha \) (\( \alpha \) list)

codatatype \( \alpha \) tree = Node \( \alpha \) (\( \alpha \) tree list)

• Compositionality = no unfolding
datatype $\alpha$ list = Nil | Cons $\alpha$ ($\alpha$ list)

codatatype $\alpha$ tree = Node $\alpha$ ($\alpha$ tree fset)

- Compositionality = no unfolding
- Need abstract interface
datatypen \( \alpha \text{ list} \) := Nil | Cons \( \alpha \) (\( \alpha \text{ list} \))

codatatypen \( \alpha \text{ tree} \) := Node \( \alpha \) (\( \alpha \text{ tree} \) fset)

- Compositionality = no unfolding
- Need abstract interface
- What interface?
Type constructors are not just operators on types!
The interface: **bounded natural functor**

- type constructor $F$
The interface: **bounded natural functor**

- type constructor $F$
- $F\text{map}$

$\text{BNF} = \text{type constructor } F \cup \text{functor } F\text{map}$
The interface: bounded natural functor

- type constructor $F$
- $F\text{map}$
- $F\text{set}$

functor

natural transformation
The interface: **bounded natural functor**

- type constructor \( F \)
- \( \text{Fmap} \)
- \( \text{Fset} \)
- \( \text{Fbd} \)

functor

natural transformation

infinite cardinal
The interface: bounded natural functor

- type constructor $F$
- $F\text{map}$
- $F\text{set}$
- $F\text{bd}$

$\text{BNF} = \text{type constructor} + \text{polymorphic constraints} + \text{assumptions}$
Type constructors are functors

\[ \text{Fmap} : (\alpha \rightarrow \alpha') \rightarrow (\beta \rightarrow \beta') \rightarrow (\alpha, \beta) \text{ F} \rightarrow (\alpha', \beta') \text{ F} \]
Type constructors are functors

\[ \text{Fmap} : (\alpha \to \alpha') \to (\beta \to \beta') \to (\alpha, \beta) \text{ F} \to (\alpha', \beta') \text{ F} \]

\[ \text{Fmap} \ id \ id \ = \ id \]

\[ \text{Fmap} \ f_1 f_2 \circ \text{Fmap} \ g_1 g_2 \ = \ \text{Fmap} \ (f_1 \circ g_1) \ (f_2 \circ g_2) \]
Type constructors are containers

\[ Fset_1 : (\alpha, \beta) \mathbf{F} \rightarrow \alpha \text{ set} \]

\[ Fset_2 : (\alpha, \beta) \mathbf{F} \rightarrow \beta \text{ set} \]
Type constructors are containers

\[ Fset_1 : (\alpha, \beta) \ F \rightarrow \alpha \ \text{set} \]
\[ Fset_2 : (\alpha, \beta) \ F \rightarrow \beta \ \text{set} \]

\[ Fset_1 \circ \text{Fmap} \ f_1 \ f_2 = \text{image} \ f_1 \circ Fset_1 \]
\[ Fset_2 \circ \text{Fmap} \ f_1 \ f_2 = \text{image} \ f_2 \circ Fset_2 \]
Further BNF assumptions

\[ \forall x \in F\text{set}_1 \ z. \ f_1 x = g_1 x \]
\[ \forall x \in F\text{set}_2 \ z. \ f_2 x = g_2 x \]
\[ \implies \ F\text{map} \ f_1 \ f_2 \ z = F\text{map} \ g_1 \ g_2 \ z \]
Further BNF assumptions

\[ \forall x \in Fset_1 \ z. \ f_1 \ x = g_1 \ x \tag{1} \]
\[ \forall x \in Fset_2 \ z. \ f_2 \ x = g_2 \ x \tag{2} \]
\[ \Rightarrow \quad \text{Fmap} \ f_1 \ f_2 \ z = \text{Fmap} \ g_1 \ g_2 \ z \]

\[ \aleph_0 \leq \text{Fbd} \]
Further BNF assumptions

∀x ∈ Fset₁ z. f₁ x = g₁ x \\
∀x ∈ Fset₂ z. f₂ x = g₂ x \Rightarrow \quad \text{Fmap } f_1 \ f_2 \ z = \text{Fmap } g_1 \ g_2 \ z

\aleph_0 \leq \text{Fbd}

|Fset_i \ z| \leq \text{Fbd}
Further BNF assumptions

\[
\forall x \in \text{Fset}_1 \ z. \ f_1 \ x = g_1 \ x \quad \forall x \in \text{Fset}_2 \ z. \ f_2 \ x = g_2 \ x \quad \Rightarrow \quad \text{Fmap} \ f_1 \ f_2 \ z = \text{Fmap} \ g_1 \ g_2 \ z
\]

\[
\aleph_0 \ \leq \ \text{Fbd}
\]

\[
|\text{Fset}_i \ z| \ \leq \ \text{Fbd}
\]

\[
|(\alpha_1, \alpha_2) \ F| \ \leq \ (|\alpha_1| + |\alpha_2|)^{\text{Fbd}}
\]
Further BNF assumptions

\( \forall x \in Fset_1 \ z. \ f_1 \ x = g_1 \ x \ \}
\( \forall x \in Fset_2 \ z. \ f_2 \ x = g_2 \ x \ \} \implies Fmap \ f_1 \ f_2 \ z = Fmap \ g_1 \ g_2 \ z \)

\( \aleph_0 \leq Fbd \)

\( |Fset_i \ z| \leq Fbd \)

\( |(\alpha_1, \alpha_2) \ F| \leq (|\alpha_1| + |\alpha_2|)^{Fbd} \)

\((F, Fmap)\) preserves weak pullbacks
What are bounded natural functors good for?

BNFs ...
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BNFs ...

- cover basic type constructors (e.g. +, ×, unit, and \( \alpha \to \beta \) for fixed \( \alpha \))
What are bounded natural functors good for?

BNFs ...

- cover basic type constructors (e.g. $+$, $\times$, unit, and $\alpha \to \beta$ for fixed $\alpha$)
- cover non-free type constructors (e.g. fset, cset)
What are bounded natural functors good for?

BNFs ...

- cover basic type constructors (e.g. $+$, $\times$, unit, and $\alpha \rightarrow \beta$ for fixed $\alpha$)
- cover non-free type constructors (e.g. fset, cset)
- are closed under composition
What are bounded natural functors good for?

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- admit initial algebras (datatypes)
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- admit initial algebras (datatypes)
- admit final coalgebras (codatatypes)
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BNFs ...

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- cover non-free type constructors (e.g. fset, cset)
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- admit initial algebras (datatypes)
- admit final coalgebras (codatatypes)
- are closed under initial algebras and final coalgebras
What are bounded natural functors good for?

BNFs ...

- cover basic type constructors (e.g. $+$, $\times$, unit, and $\alpha \rightarrow \beta$ for fixed $\alpha$)
- cover non-free type constructors (e.g. fset, cset)
- are closed under composition
- admit initial algebras (datatypes)
- admit final coalgebras (codatatypes)
- are closed under initial algebras and final coalgebras
- make initial algebras and final coalgebras **expressible** in HOL
Outline

Datatypes in HOL—State of the Art

Bounded Natural Functors

(Co)datatypes

(Co)nclusion
From user specifications to (co)datatypes

Given

```
datatype α list = Nil | Cons α (α list)
```
From user specifications to (co)datatypes

Given

```plaintext
datatype α list = Nil | Cons α (α list)
```

1. Abstract to $β = \text{unit} + α \times β$
From user specifications to (co)datatypes

Given

datatype $\alpha$ list = Nil | Cons $\alpha$ ($\alpha$ list)

1. Abstract to $\beta = \text{unit} + \alpha \times \beta$
2. Prove that $(\alpha, \beta) F = \text{unit} + \alpha \times \beta$ is a BNF
From user specifications to (co)datatypes

Given

\[
\text{datatype } \alpha \text{ list } = \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list})
\]

1. Abstract to \( \beta = \text{unit} + \alpha \times \beta \)
2. Prove that \((\alpha, \beta) \ F = \text{unit} + \alpha \times \beta\) is a BNF
3. Define F-algebras
From user specifications to (co)datatypes

Given

datatype $\alpha$ list $= \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list})$

1. Abstract to $\beta = \text{unit} + \alpha \times \beta$
2. Prove that $(\alpha, \beta)$ $F = \text{unit} + \alpha \times \beta$ is a BNF
3. Define $F$-algebras
4. Construct initial algebra

$$(\alpha \text{ list}, \text{fld} : \text{unit} + \alpha \times \alpha \text{ list} \rightarrow \alpha \text{ list})$$
From user specifications to (co)datatypes

Given

\[
\text{datatype } \alpha \text{ list } = \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list})
\]

1. Abstract to \( \beta = \text{unit} + \alpha \times \beta \)
2. Prove that \((\alpha, \beta)F = \text{unit} + \alpha \times \beta\) is a BNF
3. Define \(F\)-algebras
4. Construct initial algebra
   \[
   (\alpha \text{ list}, \text{fld} : \text{unit} + \alpha \times \alpha \text{ list} \rightarrow \alpha \text{ list})
   \]
5. Define iterator
   \[
   \text{iter} : (\text{unit} + \alpha \times \alpha \text{ list} \rightarrow \beta) \rightarrow \alpha \text{ list} \rightarrow \beta
   \]
From user specifications to (co)datatypes

Given

datatype $\alpha$ list = Nil | Cons $\alpha$ ($\alpha$ list)

1. Abstract to $\beta$ = unit + $\alpha \times \beta$
2. Prove that $(\alpha, \beta) F = \text{unit} + \alpha \times \beta$ is a BNF
3. Define F-algebras
4. Construct initial algebra

   $$(\alpha$ list, fld : unit + $\alpha \times \alpha$ list $\rightarrow$ $\alpha$ list)$$

5. Define iterator

   $\text{iter} : (\text{unit} + \alpha \times \alpha$ list $\rightarrow \beta) \rightarrow \alpha$ list $\rightarrow \beta$

6. Prove characteristic theorems (e.g. induction)
From user specifications to (co)datatypes

Given

\[ \text{datatype } \alpha \text{ list } = \text{Nil} \mid \text{Cons } \alpha (\alpha \text{ list}) \]

1. Abstract to \( \beta = \text{unit} + \alpha \times \beta \)
2. Prove that \((\alpha, \beta) F = \text{unit} + \alpha \times \beta\) is a BNF
3. Define F-algebras
4. Construct initial algebra
   \[ (\alpha \text{ list}, \text{fld} : \text{unit} + \alpha \times \alpha \text{ list} \rightarrow \alpha \text{ list}) \]
5. Define iterator
   \[ \text{iter} : (\text{unit} + \alpha \times \alpha \text{ list} \rightarrow \beta) \rightarrow \alpha \text{ list} \rightarrow \beta \]
6. Prove characteristic theorems (e.g. induction)
7. Prove that list is a BNF
From user specifications to (co)datatypes

Given

```plaintext
datatype α list = Nil | Cons α (α list)
```

1. Abstract to β = unit + α × β
2. Prove that (α, β) F = unit + α × β is a BNF
3. Define F-algebras
4. Construct initial algebra

```
(α list, fld : unit + α × α list → α list)
```

5. Define iterator

```
iter : (unit + α × α list → β) → α list → β
```

6. Prove characteristic theorems (e.g. induction)
7. Prove that list is a BNF (enables nested recursion)
From user specifications to (co)datatypes

Given

\[
\text{codatatype } \alpha \text{ llist } = \text{LNil} | \text{LCons } \alpha (\alpha \text{ llist})
\]

1. Abstract to \( \beta = \text{unit} + \alpha \times \beta \)
2. Prove that \((\alpha, \beta) F = \text{unit} + \alpha \times \beta\) is a BNF
3. Define F-coalgebras
4. Construct final coalgebra

\[
(\alpha \text{ llist}, \text{unf} : \alpha \text{ llist} \to \text{unit} + \alpha \times \alpha \text{ llist})
\]

5. Define coiterator

\[
\text{coiter : } (\beta \to \text{unit} + \alpha \times \alpha \text{ llist}) \to \beta \to \alpha \text{ llist}
\]

6. Prove characteristic theorems (e.g. coinduction)
7. Prove that \text{llist} is a BNF (enables nested corecursion)
Induction

\[ \beta = (\alpha, \beta) F \]

- Given \( \varphi : \alpha \text{ IF} \rightarrow \text{bool} \)
Induction

\[ \beta = (\alpha, \beta) F \]

- Given \( \varphi : \alpha \text{ IF} \rightarrow \text{bool} \)
- Abstract induction principle

\[ \forall z. (\forall x \in \text{Fset}_2 z. \varphi x) \Rightarrow \varphi (\text{fld} z) \]

\[ \forall x. \varphi x \]
Induction

\[ \beta = \text{unit} + \alpha \times \beta \]

- Given \( \varphi : \alpha \text{ IF} \rightarrow \text{bool} \)
- Abstract induction principle

\[
\forall z. (\forall x \in \text{Fset}_2 z. \varphi x) \Rightarrow \varphi (\text{fld } z)
\]

\[
\forall x. \varphi x
\]

- Given \( \varphi : \alpha \text{ list} \rightarrow \text{bool} \)
- Case distinction on \( z \)

\[
(\forall y s \in \emptyset. \varphi y s) \Rightarrow \varphi (\text{fld } (\text{Inl } ()))
\]

\[
\forall x s. (\forall y s \in \{xs\}. \varphi y s) \Rightarrow \varphi (\text{fld } (\text{Inr } (x, xs)))
\]

\[
\forall x s. \varphi x s
\]

\[
\forall x s. \varphi x s
\]
Induction

\( \beta = \text{unit} + \alpha \times \beta \)

- Given \( \varphi : \alpha \text{ IF} \rightarrow \text{bool} \)
- Abstract induction principle

\[ \forall z. \left( \forall x \in \text{Fset}_2 \ z. \ \varphi \ x \right) \Rightarrow \varphi \ (\text{fld} \ z) \]

\[ \forall x. \ \varphi \ x \]

- Given \( \varphi : \alpha \text{ list} \rightarrow \text{bool} \)
- Concrete induction principle

\[ \forall x \ xs. \ \varphi \ xs \Rightarrow \varphi \ (\text{fld} \ (\text{Inr} \ (x, \ xs))) \]

\[ \forall xs. \ \varphi \ xs \]
Induction

\[ \beta = \text{unit} + \alpha \times \beta \]

- Given \( \varphi : \alpha \text{ IF} \to \text{bool} \)
- Abstract induction principle

\[
\forall z. \left( \forall x \in \text{Fset}_2 z. \varphi x \right) \Rightarrow \varphi (\text{fld } z) \\
\forall x. \varphi x
\]

- Given \( \varphi : \alpha \text{ list} \to \text{bool} \)
- In constructor notation

\[
\forall x \, xs. \quad \varphi \text{ Nil} \\
\forall x \, xs. \quad \varphi \text{ xs} \Rightarrow \varphi (\text{Cons } x \, xs) \\
\forall xs. \varphi \text{ xs}
\]
Induction & Coinduction

\[ \beta = (\alpha, \beta) F \]

- Given \( \varphi : \alpha \text{IF} \rightarrow \text{bool} \)
  
- Abstract induction principle

\[
\forall z. \left( \forall x \in \text{Fset}_2 z. \varphi x \right) \Rightarrow \varphi (\text{fld} z)
\]

\[
\forall x. \varphi x
\]

- Given \( \psi : \alpha \text{JF} \rightarrow \alpha \text{JF} \rightarrow \text{bool} \)
Induction & Coinduction

\( \beta = (\alpha, \beta) F \)

- Given \( \varphi : \alpha \text{ IF} \rightarrow \text{bool} \)
- Abstract induction principle

\[ \forall z. (\forall x \in \text{Fset}_2 z. \varphi x) \Rightarrow \varphi (\text{fld} z) \]
\[ \forall x. \varphi x \]

- Given \( \psi : \alpha \text{ JF} \rightarrow \alpha \text{ JF} \rightarrow \text{bool} \)
- Abstract coinduction principle

\[ \forall x \ y. \psi x \ y \Rightarrow \text{Fpred Eq} \ \psi (\text{unf} \ x) \ (\text{unf} \ y) \]
\[ \forall x \ y. \psi x \ y \Rightarrow x = y \]
Example

codatatype $\alpha$ tree = Node (lab: $\alpha$) (sub: $\alpha$ tree fset)
Example

codatatype $\alpha$ tree = Node (lab: $\alpha$) (sub: $\alpha$ tree fset)

corec tmap : ($\alpha \rightarrow \beta$) $\rightarrow$ $\alpha$ tree $\rightarrow$ $\beta$ tree where

lab (tmap $f$ $t$) = $f$ (lab $t$)
sub (tmap $f$ $t$) = image (tmap $f$) (sub $t$)
Example

\[
\begin{align*}
\text{codatatype } \alpha \text{ tree } &= \text{Node (lab: } \alpha \text{) (sub: } \alpha \text{ tree fset)} \\
\text{corec } tmap : (\alpha \rightarrow \beta) \rightarrow \alpha \text{ tree } \rightarrow \beta \text{ tree where} \\
\text{lab } (tmap f t) &= f (\text{lab } t) \\
\text{sub } (tmap f t) &= \text{image } (tmap f) (\text{sub } t)
\end{align*}
\]

\[
\text{lemma } tmap (f \circ g) t = tmap f (tmap g t)
\]
Example

codatatype $\alpha$ tree = Node ($\text{lab}: \alpha$) ($\text{sub}: \alpha$ tree fset)

corec $tmap : (\alpha \rightarrow \beta) \rightarrow \alpha$ tree $\rightarrow \beta$ tree where

lab ($tmap f t$) = $f$ (lab $t$)  
sub ($tmap f t$) = image ($tmap f$) (sub $t$)

lemma $tmap (f \circ g) t = tmap f (tmap g t)$

by (intro tree_coinduct [where $\psi = \lambda t_1. \exists t_2. t_1 = tmap (f \circ g) t \land t_2 = tmap f (tmap g t)]))$ force+
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Category Theory Applied to Theorem Proving

- Framework for defining types in HOL
Foundational, Compositional (Co)datatypes for Higher-Order Logic

Category Theory Applied to Theorem Proving

- Framework for defining types in HOL
- Characteristic theorems are derived, not stated as axioms
Foundational, Compositional (Co)datatypes for Higher-Order Logic
Category Theory Applied to Theorem Proving

- Framework for defining types in HOL
- Characteristic theorems are derived, not stated as axioms
- Mutual and nested combinations of (co)datatypes and custom BNFs
Foundational, Compositional (Co)datatypes for Higher-Order Logic

Category Theory Applied to Theorem Proving

- Framework for defining types in HOL
- Characteristic theorems are derived, not stated as axioms
- Mutual and nested combinations of (co)datatypes and custom BNFs
- Adapt insights from category theory to HOL's restrictive type system
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Category Theory Applied to Theorem Proving

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- Adapt insights from category theory to HOL’s restrictive type system

- Formalized & implemented in Isabelle/HOL
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- Mutual and nested combinations of (co)datatypes and custom BNFs
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Thank you for your attention!
Foundational, Compositional (Co)datatypes for Higher-Order Logic
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Outline

Backup slides
Type constructors act on sets

\[(A_1, A_2) \ F = \{ z \mid \text{Fset}_1 \ z \subseteq A_1 \land \text{Fset}_2 \ z \subseteq A_2 \}\]
Type constructors act on sets

\[
(A_1, A_2) \ F = \{ z \mid F\text{set}_1 \ z \subseteq A_1 \land F\text{set}_2 \ z \subseteq A_2 \}
\]

\[
(A_1, A_2) \ F : (\alpha, \beta) \ F \text{ set}
\]

\[
(\forall i \in \{1, 2\}. \ \forall x \in F\text{set}_i \ z . \ f_i \ x = g_i \ x) \ \Rightarrow \ F\text{map} \ f_1 \ f_2 \ z = F\text{map} \ g_1 \ g_2 \ z
\]
Type constructors are bounded

Fbd: infinite cardinal
Type constructors are bounded

Fbd: infinite cardinal

|Fset_i z| \leq Fbd
Type constructors are bounded

$F \text{bd}: \text{infinite cardinal}$

$F : (\alpha, \beta) \text{ set}$

$|F \text{set}_i \ z| \leq F \text{bd}$
Type constructors are bounded

Fbd: infinite cardinal

\[ (\alpha, \beta) \mathcal{F} \]

\[ \alpha \text{ set} \quad \beta \text{ set} \]

\[ \mathcal{F} \text{ set} \]

\[ |\mathcal{F}\text{set}_i z| \leq \text{Fbd} \]

\[ |(A_1, A_2) \mathcal{F}| \leq (|A_1| + |A_2| + 2)^{\text{Fbd}} \]
Algebras, Coalgebras & Morphisms

\[ \beta = (\alpha, \beta) F \]
Algebras, Coalgebras & Morphisms

\[ \beta = (\alpha, \beta)_F \]

\[
\begin{array}{ccc}
(\alpha, A)_F & \xrightarrow{\text{Fmap id } f} & (\alpha, B)_F \\
\downarrow s & & \downarrow s_B \\
A & \xrightarrow{f} & B
\end{array}
\]
Algebras, Coalgebras & Morphisms

\[ \beta = (\alpha, \beta) F \]

\[
\begin{array}{c}
(\alpha, A) F \\
\downarrow s \\
A \\
\end{array}
\]

\[
\begin{array}{c}
A \\
\downarrow s \\
(\alpha, A) F \\
\end{array}
\]

\[
\begin{array}{c}
(\alpha, A) F \\
\xrightarrow{\text{Fmap id } f} \\
(\alpha, B) F \\
\end{array}
\]

\[
\begin{array}{c}
A \\
\downarrow S_A \\
(\alpha, A) F \\
\end{array}
\]

\[
\begin{array}{c}
B \\
\downarrow S_B \\
(\alpha, B) F \\
\end{array}
\]

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\end{array}
\]

\[
\begin{array}{c}
(\alpha, A) F \\
\xrightarrow{\text{Fmap id } f} \\
(\alpha, B) F \\
\end{array}
\]
Algebras, Coalgebras & Morphisms

\[ \beta = (\alpha, \beta) F \]

\[
\begin{array}{cc}
(\alpha, A) F & \xrightarrow{\text{Fmap id } f} & (\alpha, B) F \\
\downarrow s & & \downarrow s \\
A & & (\alpha, A) F
\end{array}
\]

\[
\begin{array}{cc}
A & \xrightarrow{f} & B \\
\downarrow s_A & & \downarrow s_B \\
A & & B
\end{array}
\]
Initial Algebras & Final Coalgebras

\[ \beta = (\alpha, \beta) F \]

- **weakly initial:** exists morphism to any other algebra
- **initial:** exists *unique* morphism to any other algebra
- **weakly final:** exists morphism from any other coalgebra
- **final:** exists *unique* morphism from any other coalgebra
Initial Algebras & Final Coalgebras

\[ \beta = (\alpha, \beta) F \]

**weakly initial:** exists morphism to any other algebra

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- Product of all algebras is weakly initial
- Suffices to consider algebras over types of certain cardinality
- Minimal subalgebra of weakly initial algebra is initial
Initial Algebras & Final Coalgebras

\[ \beta = \left( \alpha, \beta \right)_F \]

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- Product of all algebras is weakly initial
- Suffices to consider algebras over types of certain cardinality
- Minimal subalgebra of weakly initial algebra is initial
- Construct minimal subalgebra from below by transfinite recursion

\[ \Rightarrow \text{Have a bound for its cardinality} \]

\[ \Rightarrow (\alpha_{\text{IF}}, \text{fld} : (\alpha, \alpha_{\text{IF}})_F \rightarrow \alpha_{\text{IF}}) \]
Initial Algebras & Final Coalgebras

\[ \beta = (\alpha, \beta) F \]

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- **weakly final:** exists morphism from any other coalgebra
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- Suffices to consider algebras over types of certain cardinality
- Minimal subalgebra of weakly initial algebra is initial
- Construct minimal subalgebra from below by transfinite recursion
  \[ \Rightarrow \text{ Have a bound for its cardinality} \]
- Sum of all coalgebras is weakly final
- Suffices to consider coalgebras over types of certain cardinality
- Quotient of weakly final coalgebra to the greatest bisimulation is final
  \[ \Rightarrow (\alpha \text{ IF}, \text{fld} : (\alpha, \alpha \text{ IF}) F \rightarrow \alpha \text{ IF}) \]
Initial Algebras & Final Coalgebras

\[ \beta = (\alpha, \beta) F \]

**weakly initial:** exists morphism to any other algebra

**initial:** exists *unique* morphism to any other algebra

**weakly final:** exists morphism from any other coalgebra

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- Product of all algebras is weakly initial
- Suffices to consider algebras over types of certain cardinality
- Minimal subalgebra of weakly initial algebra is initial
- Construct minimal subalgebra from below by transfinite recursion
  \[ \Rightarrow \text{Have a bound for its cardinality} \]
  \[ (\alpha \text{ IF}, \text{fld} : (\alpha, \alpha \text{ IF}) F \rightarrow \alpha \text{ IF}) \]

- Sum of all coalgebras is weakly final
- Suffices to consider coalgebras over types of certain cardinality
- Quotient of weakly final coalgebra to the greatest bisimulation is final
- Use concrete weakly final coalgebra (elements are tree-like structures)
  \[ \Rightarrow \text{Have a bound for its cardinality} \]
  \[ (\alpha \text{ JF}, \text{unf} : \alpha \text{ JF} \rightarrow (\alpha, \alpha \text{ JF}) F) \]
Iteration & Coiteration

\[ \beta = (\alpha, \beta) \mathbf{F} \]

- Given \( s : (\alpha, \beta) \mathbf{F} \rightarrow \beta \)
Iteration & Coiteration

\[ \beta = (\alpha, \beta) \, F \]

- Given \( s : (\alpha, \beta) \, F \to \beta \)
- Obtain unique morphism \( \text{iter } s \) from \( (\alpha \, \text{IF}, \text{fld}) \) to \( (U_\beta, s) \)

\[(\alpha, \alpha \, \text{IF}) \, F \xrightarrow{\text{Fmap id (iter } s\text{)}} (\alpha, \beta) \, F\]

\[
\begin{array}{c}
\alpha \, \text{IF} \xrightarrow{\text{fld}} (\alpha, \alpha \, \text{IF}) \, F \\
\downarrow \text{iter } s \quad \quad \quad \quad \quad \downarrow s
\end{array}
\]

\[
\begin{array}{c}
\alpha \, \text{IF} \xrightarrow{\text{iter } s} \beta
\end{array}
\]
Iteration & Coiteration

\[ \beta = (\alpha, \beta) \mathcal{F} \]

- Given \( s : (\alpha, \beta) \mathcal{F} \to \beta \)
- Obtain unique morphism \( \text{iter } s \) from \((\alpha \text{ IF, fld)}\) to \((U\beta, s)\)

- Given \( s : \beta \to (\alpha, \beta) \mathcal{F} \)

\[
\begin{array}{ccc}
(\alpha, \alpha \text{ IF}) \mathcal{F} & \xrightarrow{\text{Fmap id (iter s)}} & (\alpha, \beta) \mathcal{F} \\
\downarrow \text{fld} & & \downarrow s \\
\alpha \text{ IF} & \xrightarrow{\text{iter s}} & \beta
\end{array}
\]
Iteration & Coiteration

\[ \beta = (\alpha, \beta) \ F \]

- Given \( s : (\alpha, \beta) \ F \to \beta \)
- Obtain unique morphism \( \text{iter } s \) from \((\alpha \ \text{IF}, \text{fld})\) to \((U\beta, s)\)

\[
\begin{align*}
(\alpha, \alpha \ \text{IF}) \ F & \xrightarrow{\text{Fmap id (iter } s\text{)}} (\alpha, \beta) \ F \\
\alpha \ \text{IF} & \xrightarrow{\text{fld}} \alpha \ \text{IF} & \xrightarrow{\text{iter } s} \beta
\end{align*}
\]

- Given \( s : \beta \to (\alpha, \beta) \ F \)
- Obtain unique morphism \( \text{coiter } s \) from \((U\beta, s)\) to \((\alpha \ \text{JF}, \text{unf})\)

\[
\begin{align*}
\beta & \xrightarrow{\text{coiter } s} \alpha \ \text{IF} \\
(\alpha, \beta) \ F & \xrightarrow{\text{Fmap id (coiter } s\text{)}} (\alpha, \alpha \ \text{IF}) \ F
\end{align*}
\]
Preservation of BNF Properties

\[ \beta = (\alpha, \beta) F \]

- \( \text{IFmap } f = \text{iter } (\text{fld } \circ \text{ Fmap } f \text{ id}) \)
- \( \text{IFset } = \text{iter } \text{collect}, \text{ where} \)

\[
\text{collect } z = Fset_1 \, z \cup \bigcup Fset_2 \, z
\]
Preservation of BNF Properties

\[ \beta = (\alpha, \beta) \hat{F} \]

- IFmap \( f \) = iter (fld \circ Fmap f id)
- IFset = iter collect, where

\[ \text{collect } z = \text{Fset}_1 z \cup \bigcup \text{Fset}_2 z \]

**Theorem**

(IF, IFmap, IFset, \( 2^{Fbd} \)) is a BNF
Preservation of BNF Properties

\[ \beta = (\alpha, \beta) F \]

- IFmap \( f = \text{iter (fld } \circ \text{ Fmap } f \text{ id)} \)
- IFset = \text{iter collect, where}
  
  \[ \text{collect } z = \text{Fset}_1 z \cup \bigcup \text{Fset}_2 z \]

- JFmap \( f = \text{coiter (Fmap } f \text{ id } \circ \text{ unf)} \)
- JFset \( x = \bigcup_{i \in \mathbb{N}} \text{collect}_i x, \text{ where}
  
  \[ \text{collect}_0 x = \emptyset \]
  \[ \text{collect}_{i+1} x = \text{Fset}_1 (\text{unf } x) \cup \bigcup \text{collect}_i y \]
  \[ y \in \text{Fset}_2 (\text{unf } x) \]

\textbf{Theorem}

(IF, IFmap, IFset, \(2^{\text{Fbd}}\)) is a BNF
Preservation of BNF Properties

\[ \beta = (\alpha, \beta) F \]

- \( \text{IFmap } f = \text{iter } (\text{fld } \circ \text{Fmap } f \text{ id}) \)
- \( \text{IFset } = \text{iter } \text{collect}, \text{ where} \)
  \[ \text{collect } z = \text{Fset}_1 z \cup \bigcup \text{Fset}_2 z \]

- \( \text{JFmap } f = \text{coiter } (\text{Fmap } f \text{ id } \circ \text{unf}) \)
- \( \text{JFset } x = \bigcup_{i \in \mathbb{N}} \text{collect}_i x, \text{ where} \)
  \[ \text{collect}_0 x = \emptyset \]
  \[ \text{collect}_{i+1} x = \text{Fset}_1 (\text{unf } x) \cup \bigcup \text{collect}_i y \]
  \[ y \in \text{Fset}_2 (\text{unf } x) \]

**Theorem**

(\( \text{IF, IFmap, IFset, } 2^{\text{Fbd}} \)) is a BNF

**Theorem**

(\( \text{JF, JFmap, JFset, } \text{Fbd}^{\text{Fbd}} \)) is a BNF