PERSONAL - Data Structures and Algorithms

- **EDITOR'S NOTE**: I *highly* recommend watching videos/demos of these topics and/or playing around with these algorithms and concepts yourself, the definitions and examples in text form won't be enough to fully understand the concept behind these algorithms - they are merely here to provide a rough blueprint
    - try and do the tutorial exerices, solve old exams or use a tool like TUMGAD to automatically generate exercises and corresponding solutions

Complexity

- **Time Complexity**: relation between **growth of runtime** and **growth of input**
    - I: **set of instances** of an algorithm with different inputs
    - $T : I \to \mathbb{N}$: **runtime** of an algorithm, usually measured in an actual unit of time (*e.g. nanoseconds*)
- **Space Complexity**: relation between **used space / memory** and **growth of input**

Different Cases

- I_n: set of instances of size n
- **worst case**: $t(n) = \max\{T(i) : i \in I_n\}$
    - pessimistic, but guarantees efficiency
- **average case**: $t(n) = \frac{1}{|I_n|} \sum_{i \in I_n} T(i)$ if $I_n \in \mathbb{N}$, else $t(n) = \sum_{i \in I_n} P[i] \cdot T(i)$
    - average, but doesn't necessarily define usual behavior
- **best case**: $t(n) = \min\{T(i) : i \in I_n\}$

- best result possible, but very optimistic

Landau-Notation

- **limiting behavior** when the argument **tends towards a particular value** or **infinity**
  - $g \in \mathcal{O}(f(n)) = \{g(n) \mid \exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : g(n) \leq c \cdot f(n)\}$
    - functions that do not grow **faster** asymptotically than f (*upper asymptotic bound*)
    - $g(n) \in \mathcal{O}(f(n)) \equiv g$ is **at most a positive constant multiple** of f **for all sufficiently large values of** n
  - $g \in o(f(n)) = \{g(n) \mid \forall C > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : g(n) \leq C \cdot f(n)\}$
    - functions that grow **slower** than f
    - $o(f(n)) \subseteq \mathcal{O}(f(n))$
  - $g \in \Omega(f(n)) = \{g(n) \mid \exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : g(n) \geq c \cdot f(n)\}$
    - functions that do not grow **slower** asymptotically than f (*lower asymptotic bound*)
  - $g \in \omega(f(n)) = \{g(n) \mid \forall C > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 : g(n) \geq C \cdot f(n)\}$
    - functions that grow **faster** than f
    - $\omega(f(n)) \subseteq \Omega(f(n))$
  - $g \in \Theta(f(n)) = \mathcal{O}(f(n)) \cap \Omega(f(n))$
    - functions that have **the same growth rate** as f
    - $\Theta(f(n)) \subseteq \mathcal{O}(f(n))$ and $\Theta(f(n)) \subseteq \Omega(f(n))$
- (!) $\omega(f(n)) \cap o(f(n)) = \emptyset$
- (!) $f(n) \in o(g(n)) \implies g(n) \in \omega(f(n))$
- some **use case examples**…
  - $5n^2 - 7n \in \mathcal{O}(n^2)$, but also $\mathcal{O}(n^3), \mathcal{O}(n^4)$… (*also included in "greater" sets, since $\mathcal{O}$ defines the **upper** bound*)
  - $5n^2 - 7n \in \Omega(n^2)$, but also $\Omega(n)$ (*also included in "lesser" sets, since Ω defines the **lower** bound*)
  - $5n^2 - 7n \in \Theta(n^2)$ (*only one, since Θ defines the intersection between the two previous sets!*)
- **placeholders**:
  - instead of $g(n) \in O(f(n))$, one can write $g(n) = O(f(n))$
  - instead of $f(n) + g(n)$ for $g(n) \in o(h(n))$, one can write $f(n) + o(h(n))$
  - instead of $O(f(n)) \subseteq O(g(n))$, one can write $O(f(n)) = O(g(n))$
  - **example**: $n^3 + n = n^3 + o(n^3) = n^3(1 + o(1)) = O(n^3)$
- $\lim$-**definitions**:

- $\lim_{n\to\infty} |\frac{f(n)}{g(n)}| = 0 \implies f(n) \in o(g(n))$
- $\lim_{n\to\infty} |\frac{f(n)}{g(n)}| = 1 \implies f(n) \in \omega(g(n))$
- $\lim_{n\to\infty} |\frac{f(n)}{g(n)}| = \infty \implies f(n) \in \omega(g(n))$
- $\lim_{n\to\infty} |\frac{f(n)}{g(n)}| = c,\ 0 < c < \infty \implies f(n) \in \Theta(g(n))$

- *some more criteria*:
  - $f(n) \in O(g(n)) \iff \lim_{n\to\infty} |\frac{f(n)}{g(n)}| < \infty$
  - $f(n) \in \Omega(g(n)) \iff \lim_{n\to\infty} |\frac{f(n)}{g(n)}| > 0$
- when doing $\lim$-calculations…
  - …for proving $f(x) \in o(g(x))$ or $f(x) \in O(g(x))$, one can use $\leq$ in the proof
  - …for proving $f(x) \in \omega(g(x))$ or $f(x) \in \Omega(g(x))$, one can use $\geq$ in the proof
- **rules for $\lim$-calculations**:
  - $\lim_{n\to\infty} c \cdot f(n) = c \cdot (\lim_{n\to\infty} f(n))$
  - $\lim_{n\to\infty}(f(n) + g(n)) = \lim_{n\to\infty} f(n) + \lim_{n\to\infty} g(n)$
  - $\lim_{n\to\infty}(f(n) \cdot g(n)) = \lim_{n\to\infty} f(n) \cdot \lim_{n\to\infty} g(n)$
  - $\lim_{n\to\infty} f(n)^p = (\lim_{n\to\infty} f(n))^p$
  - $\lim_{n\to\infty} \log f(n) = \log(\lim_{n\to\infty} f(n))$
- **L'Hospital**:
  - if the result of $\lim_{n\to\infty}$ is undefined, e.g. $\frac{0}{0}$, $0 \cdot \infty$, $\infty - \infty$, $\frac{\infty}{\infty}$, 0^0 or ∞^0, use *L'Hospital's rule*
  - $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)'}{g(n)'}$
- **logarithm rules**:
  - $\ln(x \cdot y) = \ln(x) + \ln(y)$
  - $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$
  - $\ln(x^y) = y \cdot \ln(x)$
  - $\ln(e) = 1$
  - $\ln(1) = 0$
  - $\ln(\frac{1}{x}) = -\ln(x)$
  - $\log_x(y) = \frac{\log(y)}{\log(x)}$
- from **best** to **worst**…
  1. $\mathcal{O}(1)$
  2. $\mathcal{O}(\log n)$
  3. $\mathcal{O}(n)$
  4. $\mathcal{O}(n \log n)$

5. $\mathcal{O}(n^2)$
6. $\mathcal{O}(2^n)$
7. $\mathcal{O}(n!)$

- **(!)** growth rate of k**-order polynomials** $p(n) = \sum_{i=0}^{k} a_i n^i \in \Theta(n^k)$
- **properties** (valid for $\mathcal{O}$ and Ω):
  - $c \cdot f(n) \in \Theta(f(n))$ for any constant $c > 0$
  - $O(f(n)) + O(g(n)) = O(f(n) + g(n))$
  - $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$
  - $O(f(n) + g(n)) = O(f(n)) \iff g(n) \in O(f(n))$
- **properties of derivatives** (*but not the other way around!*):
  - if $f'(n) \in O(g'(n))$, then $f(n) \in O(g(n))$
  - if $f'(n) \in \Omega(g'(n))$, then $f(n) \in \Omega(g(n))$
  - if $f'(n) \in o(g'(n))$, then $f(n) \in o(g(n))$
  - if $f'(n) \in \omega(g'(n))$, then $f(n) \in \omega(g(n))$

Properties of Big O Notation

- if $f(x)$ is a **sum of several terms**, if there is one with the **largest growth rate**, **it can be kept** and **all others omitted**
  - e.g. $f(x) = 6x^4 - 2x^3 + 5 \to f(x) \in \mathcal{O}(6x^4)$
- if $f(x)$ is a **product of several factors**, any **constants** (*factors that do not depend on x*) **can be omitted**
  - e.g. $f(x) = 6x^4 \to f(x) \in \mathcal{O}(x^4)$
- if f can be written as **a finite sum of other functions**, the **fastest growing one** determines the order of f
  - e.g. $f(n) = 9\log n + 5(\log n)^4 + 3n^2 + 2n^3 \in \mathcal{O}(n^3)$
- **product**:
  - $f_1 \in \mathcal{O}(g_1) \wedge f_2 \in \mathcal{O}(g_2) \implies f_1 f_2 \in \mathcal{O}(g_1 g_2)$
  - $f \cdot \mathcal{O}(g) = \mathcal{O}(fg)$
- **sum**:
  - $f_1 \in \mathcal{O}(g_1) \wedge f_2 \in \mathcal{O}(g_2) \implies f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$
- **scalar multiplication**:
  - $\mathcal{O}(|k| \cdot g) = \mathcal{O}(g)$ for any non-zero constant k

Time Complexity

- basic **terminology**:

- linear time: $\mathcal{O}(n)$
- constant time: $\mathcal{O}(1)$
- quadratic time: $\mathcal{O}(n^2)$

- **runtime analysis** - worst case $T(I)$ for a given construct I
  - $T(\text{variable definition}) = O(1)$
  - $T(\text{comparison}) = O(1)$
  - $T(\texttt{return x}) = O(1)$
  - $T(\texttt{new Type(...)}) = O(1) + O(T(\text{constructor}))$
  - $T(I_1; I_2) = T(I_1) + T(I_2)$
  - $T(\texttt{if(C)}\ I_1\ \texttt{else}\ I_2) = O(T(C) + \max\{T(I_1), T(I_2)\})$
  - $T(\texttt{for(i = a; i < b; i++)}\ I) = O(\sum_{i=a}^{b-1}(1 + T(I)))$
  - $T(\texttt{e.m(...)}) = O(1) + T(ss)$ with ss being the body of m

Expected Values

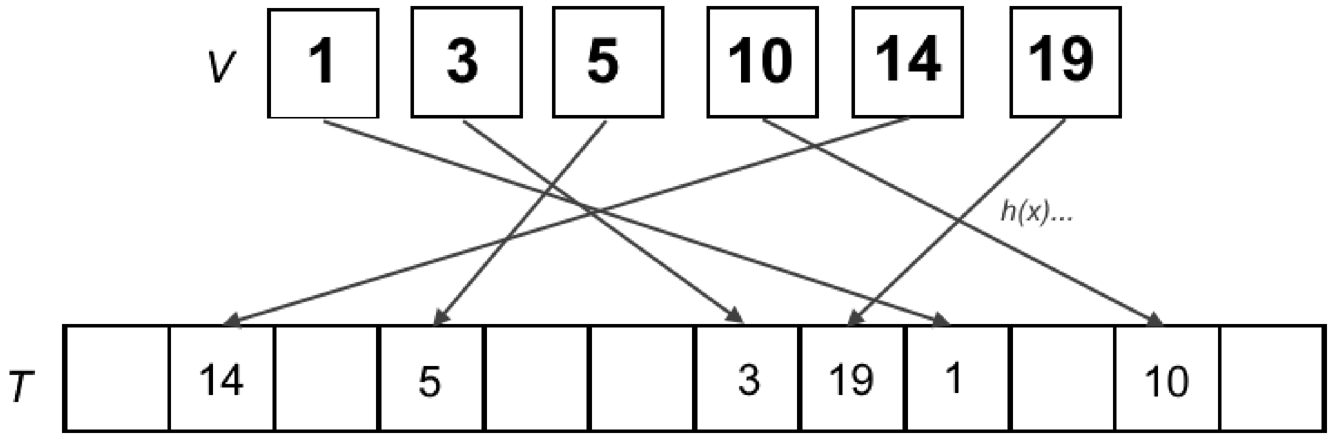
- **average case complexity**: $t(n) = \sum_{i \in I_n} p_i \cdot T(i)$
- **definitions**:
  - **sample space** Ω: set of possible results
    - e.g. single dice roll $\Omega = \{1, 2, 3, 4, 5, 6\}$, double dice roll $\Omega = \{(1, 1), (1, 2), ..., (6, 6)\}$
  - **random variable**: $X : \Omega \to \mathbb{R}$ (*variable, whose value is a random number*)
    - e.g. dice roll, X can be a number between 1 and 6
  - **domain**: $W_X := X(\Omega) = \{x \in \mathbb{R} | \exists \omega \in \Omega : X(\omega) = x\}$
    - e.g. outcome / payout in € for gambling
  - $\Pr[X = x] := \Pr[X^{-1}(x)] = \sum_{\omega \in \Omega | X(\omega) = x} \Pr[\omega]$
  - **expected value**: $\mathbb{E}[X] := \sum_{x \in W_X} x \cdot \Pr[X = x] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$
    - e.g. dice roll, $\forall x \in \{1, ..., 6\} : P[X = x] = \frac{1}{6}$...
    - ...then $\mathbb{E}[X] = \sum_x x \cdot P[X = x] = 1 \cdot P[X = 1] + 2 \cdot P[X = 2] + ... + 6 \cdot P[X = 6] = (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6} = 3.5$
  - for a **finite sample space** and **equally probable** events: $\mathbb{E}[x] = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} X(\omega)$
- **rules for expected value calculation**:
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for random variables X and Y
  - $\mathbb{E}[a \cdot X] = a \cdot \mathbb{E}[X]$ for any constant $a \in \mathbb{R}$
- **complete example**:
  - +€4 for hearts, +€7 for diamonds, -€5 for spades and -€3 for clubs

- +€1 for any ace
  - $\Omega = \{\heartsuit A, \heartsuit K, ..., \heartsuit 2, \diamondsuit A, \diamondsuit K, ..., \diamondsuit 2, \clubsuit A, \clubsuit K, ..., \clubsuit 2, \spadesuit A, \spadesuit K, ..., \spadesuit 2\}$
  - X: amount of money to get paid / pay
  - $W_X = \{-5, -4, -3, -2, 4, 5, 7, 8\}$
  - $\Pr[X = -3] = \Pr[\spadesuit K] + ... + \Pr[\spadesuit 2] = \frac{12}{52} = \frac{3}{13}$
  - $\mathbb{E}[X] = 4 \cdot \frac{12}{52} + 5 \cdot \frac{1}{52} + 7 \cdot \frac{12}{52} + 8 \cdot \frac{1}{52} + (-5) \cdot \frac{12}{52} + (-4) \cdot \frac{1}{52} + (-3) \cdot \frac{12}{52} + (-2) \cdot \frac{1}{52} = \frac{43}{52} \approx €0.83$ per card

Amortized Analysis (Accounting Method)

- **amortized analysis**: analyze costs (time, memory) of an algorithm by averaging out the worst operations over time for a sequence of n operations $(\sigma_1, ..., \sigma_n)$ (*upper bound of actual runtime* T)
  - e.g. for a dynamic array, when "pushing" a new element in a full array, the array size needs to be increased (*doubled, for the sake of simplicity*), but this only happens very rarely, so giving the method a runtime of $O(n)$ is quite pessimistic
    - instead, using amortized costs, we classify the runtime as $O(1)$, since that is what happens most of the time
- **lecture method**:
  - $S = \{\sigma_1, ..., \sigma_n\}$: set of operations
  - $T(\sigma)$: **runtime / upper bound** of an operation $\sigma \in S$
  - $T(\sigma_1, ..., \sigma_m) := \sum_{i=1}^{m} T(\sigma_i)$: **runtime / upper bound** of operation sequence $(\sigma_1, ..., \sigma_m)$
  - $\Delta(\sigma)$: **token cost** of an operation, account balance change caused by σ
    - $\Delta(\sigma) > 0$: deposit to account
    - $\Delta(\sigma) < 0$: withdraw from account
  - $A(\sigma) := T(\sigma) + \Delta(\sigma)$: **amortized runtime** of σ
    - $A(\sigma_1, ..., \sigma_m) := \sum_{i=1}^{m} A(\sigma_i)$: **amortized runtime** of operation sequence $(\sigma_1, ..., \sigma_m)$ (*upper bound of actual runtime*)
  - **accounting method**: define $\Delta : S \to \mathbb{R}$ with the following properties…
    1. $\sum_{i=1}^{m} \Delta(\sigma_i) \geq 0$ for all valid operation sequences (*i.e. no overdraft!*)
    2. Δ is chosen as fittingly as possible $\equiv$ let $A(\sigma_1, ..., \sigma_m)$ be **as small as possible**
  - 1 $\to$ account balance can **never be negative**
  - 2 $\to$ upper bound of worst-case operations is $O(m \cdot \max_{\sigma \in S}(A(\sigma)))$
  - **final step**: prove that $\sum_{i=1}^{m} \Delta(\sigma_i) \geq 0$ can never be negative
  - **hint**: look at how to calculate the actual $T(\sigma)$ function and create a $\Delta(\sigma)$ based on what the exercise wants

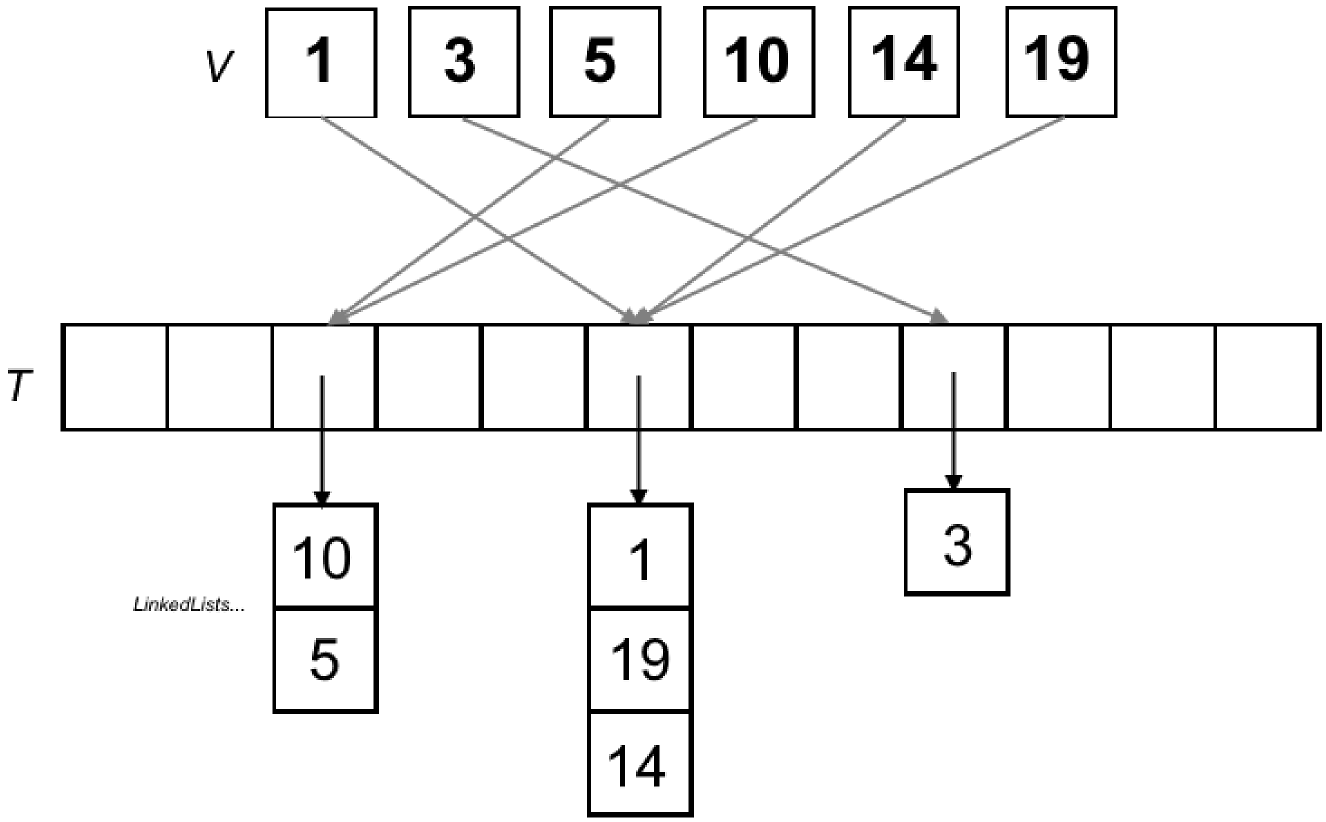
Hashing



- **map** data (*keys*) to fixed-value sizes (*values*) using a **hash function** to uniquely identify said data in a **hash table**
  - in other words, convert keys into other values and store these new values at corresponding positions inside a table
- implemented using a dictionary (*stores a set of elements where each element is identified via a unique key*)
- universe U of keys with $|U| = N$ (*some very large positive integer*)
- $V \subseteq U$: subset of actually used keys with $|V| = n$ significantly smaller than N
- **general idea**: let T be an array with space for m elements (*hash table*) and a function $h : U \to \{0, ..., m-1\}$ be used to map key to array index (*hash function*)
- **probability of hash collisions for equally spread out hash positions**: $1 - o(1)$

```
// implementing a hash table with hash function
// only for dynamic dictionaries
void insert(Object e){
    T[h(key(e))] = e;
}

// only for dynamic dictionaries
void remove(Key k){
    T[h(k)] = null;
}

// for static and dynamic dictionaries
Object find(Key k){
    return T[h(k)];
}
```

Hashing with Chaining



- **idea**: avoid collisions (*different keys mapped to the same value*) by having each cell of the hash table point to a **linked list** containing the hashed values
  - in other words, the hash table is an *array*, where each entry is a *linked list*

```
// init. array of linked lists
List<Object>[m] T;

// insert into linkedlist inside array
void insert(Object e){
    T[h(key(e))].insert(e);
}

// remove from linkedlist inside aray
void remove(Key k){
    T[h(k)].remove(k);
}

// find in linkedlist inside array
Object find(Key k){
    return T[h(k)].find(k);
}
```

- **space complexity**: $O(n+m)$
- **time complexity**:

- `insert()`: $O(1)$
- `remove()`, `find()`: $O(1 + n/m)$
  - $O(1 + c \cdot n/m)$ for c-universal hash families

Universal Hashing

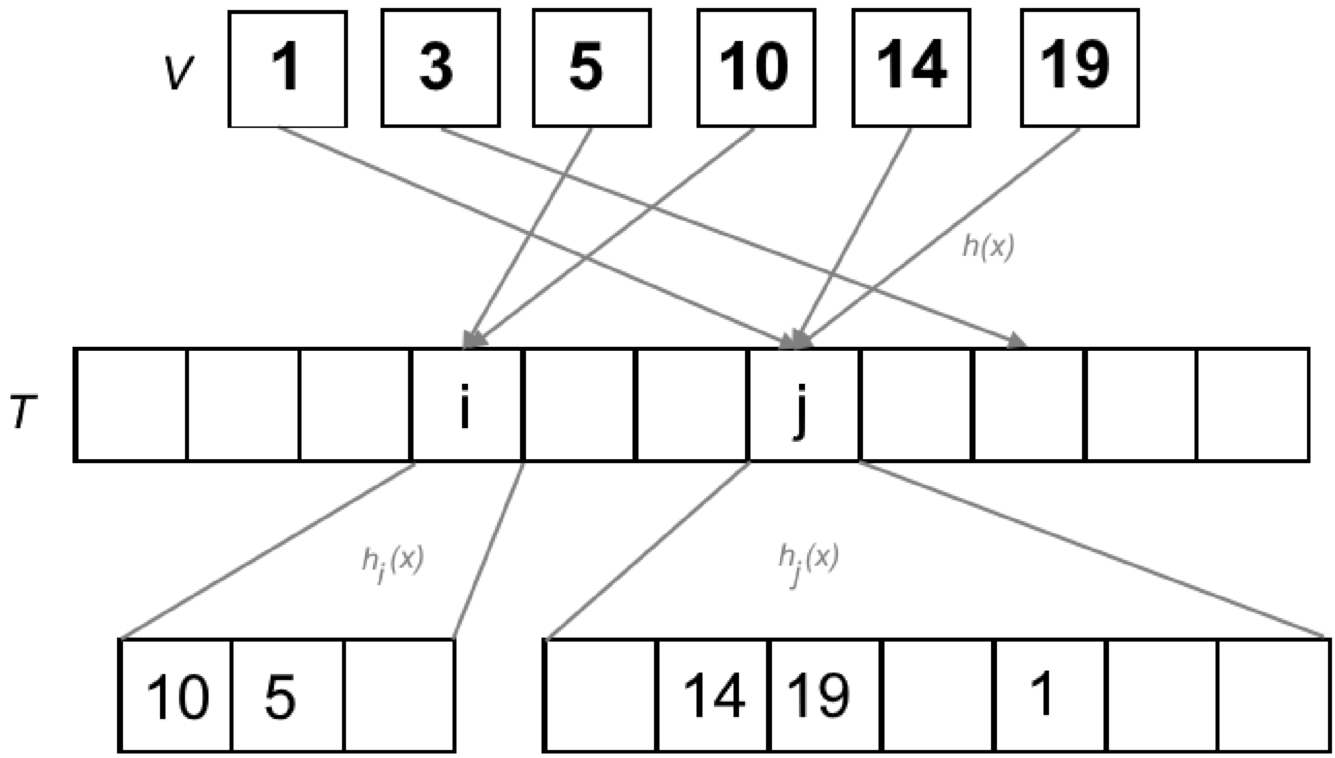
- $c > 0$: constant
- H: **family** of hash functions
- m: **size** of hash, such that each hash function returns a hash code in range $\{0, 1, ..., m-1\}$
- a family of hash functions is c-**universal**, if the probability of a hash collision between two keys x and y when *randomly* chosing a hash function $h \in H$ is less than or equal to $\frac{c}{m}$
  - *in other words*: $|\{h \in H : h(x) = h(y)\}| \leq \frac{c}{m}|H|$ for all $x \neq y$
  - *formally*: $\Pr[h(x) = h(y)] \leq \frac{c}{m}$ for all $x \neq y$
- **(1-)universal family**: c-universal family of hash functions for $c = 1$
  - *in other words*: $|\{h \in H : h(x) = h(y)\}| \leq \frac{1}{m}|H|$ for all $x \neq y$
  - *formally*: $\Pr[h(x) = h(y)] \leq \frac{1}{m}$ for all $x \neq y$
- **(*) how to determine, whether or not a given family is universal**:
  - *statement*: you are given a hash table of size m (*here* $m = 4$), a set of keys (*here A, B, C, D, E) and the given mappings for each hash function h_i
    - *example*:
      - $h_1 : A \mapsto 1, B \mapsto 1, C \mapsto 1, D \mapsto 3, E \mapsto 3$
      - $h_2 : A \mapsto 2, B \mapsto 2, C \mapsto 0, D \mapsto 0, E \mapsto 1$
      - $h_3 : A \mapsto 3, B \mapsto 1, C \mapsto 0, D \mapsto 3, E \mapsto 2$
      - $h_4 : A \mapsto 0, B \mapsto 2, C \mapsto 1, D \mapsto 2, E \mapsto 1$
      - $h_5 : A \mapsto 1, B \mapsto 3, C \mapsto 1, D \mapsto 2, E \mapsto 0$
      - $h_6 : A \mapsto 3, B \mapsto 2, C \mapsto 0, D \mapsto 1, E \mapsto 3$
  - *question*: is the hash family H_i (*here* $H_1 = \{h_1, h_2, h_4, h_5\}$) universal?
  - *answer*: prove that $|h \in H_1 : h(x) = h(y)| \leq 1$
    - *step 1*: note **collisions** between each possible pair for each hash function
      - $A/B : h_1, h_2$
      - $A/C : h_1, h_5$
      - $A/D : \emptyset$
      - $A/E : \emptyset$
      - $B/C : h_1$
      - $B/D : h_4$

        - $B/E : \emptyset$
        - $C/D : h_2$
        - $C/E : h_4$
        - $D/E : h_1$
    - *step 2*: check that the number of collisions **at most 1** for any given pair; if **not**, the function is not 1-universal, but **at least** c **universal** with c being the number of collisions
        - since $|h \in H_1 : h(A) = h(B)| = |\{h_1, h_2\}| = 2$, the family H_1 is c-universal for $c \geq 2$

- **parameterized hash families**: a hash function $h_b = b \cdot x \mod m$ defines a family of hash functions $H = \{h_b \mid b \in \mathbb{Z}\}$
    - b can be freely chosen, e.g. $h_2 = 2x \mod m$, $h_{-400} = -400x \mod m$
    - if m is a **prime number**, then $H = \{h_a : a \in \{0, ..., m-1\}^k\}$ with $h_a(x) = a \cdot x \mod m$ is a **universal family of hash functions**

- *lecture example - hashing an integer* x:
    - choose prime table size m
        - e.g. $m = 269$
    - let $w = \lfloor \log_2 m \rfloor$
        - e.g. $w = \lfloor \log_2 269 \rfloor = 8$
    - separate bitstring x (*binary representation*) into k equal parts with w bits each
        - e.g. $k = 4$, since $4 \cdot 8 = 32$
    - interpret each part as an integer $x_i \in [0, ..., 2^w - 1]$
        - e.g. $x_i \in [0, ..., 255]$ (*an unsigned byte*)
    - interpret key x to compute hash value of as k-vector of x_i, with $x = (x_1, ..., x_k)^T, x_i \in \{0, ..., 2^w - 1\}$
        - e.g. $x = (11, 7, 4, 3)^T$
    - define some vector $a = (a_1, ..., a_k)^T, a_i \in \{0, ..., m-1\}$
        - e.g. $a = (2, 4, 261, 16)^T$
    - scalar product of a and x is $a \cdot x = \sum_{i=1}^{k} a_i x_i$
    - define $h_a : x \to \{0, ..., m-1\}$ as $h_a(x) = a \cdot x \mod m$ (*scalar product of* a *and* x*, product modulo* m)
        - e.g. $h_a(x) = (2x_1 + 4x_2 + 261x_3 + 16x_4) \mod 269$
            - $h_a(46915) = (2 \cdot 11 + 4 \cdot 7 + 261 \cdot 4 + 16 \cdot 3) \mod 269 = 66$

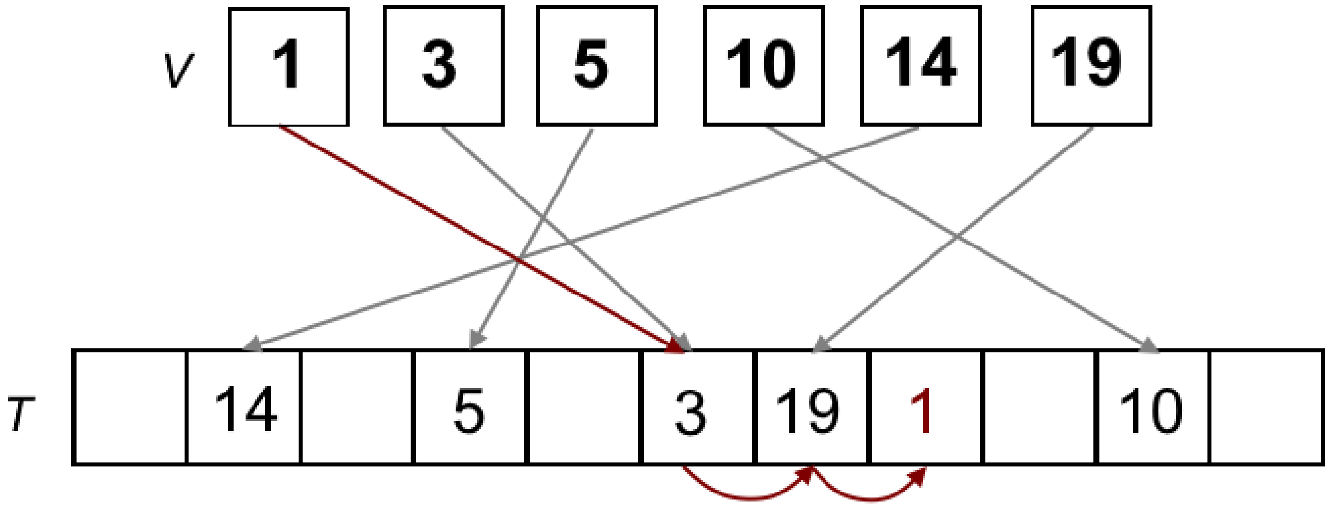
Perfect Hashing



- *“remember: no collisions.”*
- **given**:
  - **static** dictionary S of length n with keys $k_1, ..., k_n$
  - H_m: c-universal family of hash functions to $\{0, ...m-1\}$
  - $C(h)$: number of collisions in S for h for every pair (x, y)
- expected number of collisions: $E[C(h)] \leq \frac{cn(n-1)}{m}$
- for at least half of the functions, $C(n) \leq 2cn(n-1)/m$ applies
- if $m \geq cn(n-1)+1$, then at least half of the functions h in H_m are injective (no collisions)
- **strategy**: double hashing with $O(n)$ space complexity
  - **step 1**: hash key using a well-chosen hash function in a table of size $O(n)$ (i.e. $m = \alpha n$), where each collision is packed into a bucket
    - set α to $\sqrt{2} \cdot c$, then $m = \lceil \sqrt{2} \cdot c \cdot n \rceil$
    - choose h with **few collisions** from $H_{\lceil \sqrt{2} \cdot c \cdot n \rceil}$ for $h(k) \in \{0, ..., \lceil \sqrt{2}cn \rceil - 1\}$
    - choose h until $C(h) \leq \sqrt{2} \cdot n$
      - *formally*: $C(h) = |\{(x, y) \mid h(x) = h(y), x \neq y\}| \leq \sqrt{2}n$
      - *note*: every tuple pair is counted twice! (x, y), then (y, x)
    - for each hash l, a bucket B_l is created, so that each key mapped to l gets inserted into B_l
      - each bucket has $b_l = |B_l|$ keys
      - each bucket has size $m_l = c \cdot b_l(b_l - 1) + 1 \in O(b_l^2)$
        - sum of all bucket sizes effectively in $O(n)$ due to low number of collisions

        - **(!) good** function, when $\alpha \mod x \implies x > \sqrt{2} \cdot n$ for any α
    - **step 2**: choose fitting h_l for bucket B_l from c-universal family H_{m_l} with $h_l(k) \in \{0, ..., m_l - 1\}$
        - choose h_l until **no collisions**
        - **(!) good** function, when $\alpha \mod x \implies x \geq b_l(b_l - 1) + 1$ for any α with current bucket size b_l
- when using arrays, the hash value of a key x is $s_l + h_l(x)$ with $l = h(x)$, where h is a perfect hash function, and the worst-case runtime complexity for a lookup is $O(1)$

Linear Probing

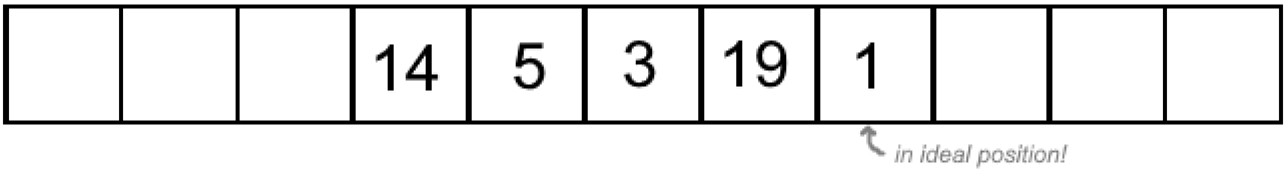


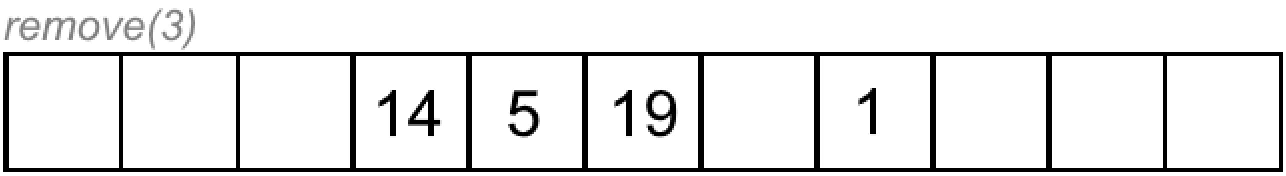
- **open** hash function, allowing for collision-causing entries to be inserted at a free neighbouring space
- **idea**: store element in *next free space*, scanning from left to right and wrapping around
    - the original hash value of a key is its **ideal position**

```
// insert into next available spot if ideal spot taken
void insert(Object e) {
    i = h(key(e));
    while (T[i] != null && T[i] != e)
        i = (i + 1) % m;
    T[i] = e;
}

// find object using linear search
Object find(Key k) {
    i = h(k);
    while (T[i] != null && key(T[i]) != k)
        i = (i+1) % m;
```

```
    return T[i];
}
```

- **pros**:
  - no extra space complexity
  - cache-efficient, since we only look at neighbouring entries in the same array
- **deletion** (*move everything **that is not on its ideal position** back one space until blank position reached, leave elements in **ideal positions** where they are!*):





- **runtime**: $O(1)$

Sorting Algorithms and their Complexities

SelectionSort

- **in-place, unstable, time complexity always $\Theta(n^2)$, space complexity $O(1)$**
- *idea*: choose **smallest element** from remainder of array and **swap places with element at start of iteration**

```
void selectionSort(Object[] a, int n) {
    for (int i = 0; i < n; i++) {
        int k = i;

        // find smallest element from unsorted sublist
        for (int j = i + 1; j < n; j++)
            if (a[j] < a[k])
                k = j;

        // swap with leftmost unsorted element
        swap(a, i, k);
    }
}
```

- *example*:

Sorted	Unsorted	Least (unsorted)
()	(*11*,25,12,22,64)	11
(*11*)	(25,*12*,22,64)	12
(11,*12*)	(25,*22*,64)	22
(11,12,*22*)	(*25*,64)	25
(11,12,22,*25*)	(*64*)	64
(11,12,22,25,*64*)	()	

InsertionSort

- **in-place, stable, worst-case** $O(n^2)$, **average-case** $O(n^2)$, **best-case** $O(n)$, **space complexity** $O(1)$
- *idea*: **take next element** from array and **insert into correct position** by **iterating backwards**

```
void insertionSort(Object[] a, int n) {
    for (int i = 1; i < n; i++)
        // iterate backwards and insert at correct position
        for (int j = i - 1; j >= 0; j--)
            if (a[j] > a[j + 1])
                swap(a, j, j + 1);
            else
                break;
}
```

- *example of a single iteration*:
  - *array*: [5,10,19,*1*,14,3]
  - *current element*: 1
    - [5,10,19,*1*,14,3] (correct position? no $\rightarrow$ swap 1 and 19)
    - [5,10,*1*,19,14,3] (correct position? no $\rightarrow$ swap 1 and 10)
    - [5,*1*,10,19,14,3] (correct position? no $\rightarrow$ swap 1 and 10)
    - [*1*,5,10,19,14,3] (correct position? yes)

MergeSort

- **not in-place, stable, worst-case** $O(n \log n)$, **average-case** $\Theta(n \log n)$, **best-case** $\Omega(n \log n)$, **space complexity** $O(n)$
- *idea*: **split** array **recursively** into two parts, then **merge** together
  - *step 1*: divide unsorted list recursively by halving it until each sublist only has one element
  - *step 2*: merge until no sublists remain, with smaller elements coming before bigger ones in each step

- *look, there's a million different implementations of MergeSort, go find one that suits you best.*

- *example*:
  - *divide*:
    - $10, 5, 7, 19, 14, 1, 3$
    - $10, 5, 7, 19 \mid 14, 1, 3$
    - $10, 5 \mid 7, 19 \mid 14, 1 \mid 3$
    - $10 \mid 5 \mid 7 \mid 19 \mid 14 \mid 1 \mid 3$
  - *conquer*:
    - $10 \mid 5 \mid 7 \mid 19 \mid 14 \mid 1 \mid 3$
    - $5, 10 \mid 7, 19 \mid 1, 14 \mid 3$
    - $5, 7, 10, 19 \mid 1, 3, 14$
    - $1, 3, 5, 7, 10, 14, 19$

QuickSort

- **in-place, unstable, worst-case** $O(n^2)$**, average-case** $O(n \log n)$**, best-case** $O(n \log n)$**, space complexity** $O(n)$
- *idea*: choose **pivot element**, then **split array** into elements **smaller** than pivot and **greater or equal** to pivot
  - for each iteration, place pivot right at the end in the beginning for simplicity's sake
  - let `itemFromLeft` be the first element starting from the left of the array that is larger than the pivot and `itemFromRight` the first element starting from the right of the array that is smaller than the pivot
  - once both have been found, swap places
  - if the index of `itemFromLeft` (i) is greater than that of `itemFromRight` (j), stop and swap pivot with `itemFromLeft`'s index
  - continue recursively for each array (*lower or greater than pivot, leave pivot unchanged in final array*)
    - *speedup*: when there are only two or less elements in an array, sort in one go without pivot element

```
void quickSort(int[] a, int l, int r) {
    if (l < r) {
        int p = a[r]; // choose rightmost element as pivot
        int i = l - 1; // left index
        int j = r; // right index
        do {
            // move left index
```

```
            do {
                i++;
            } while (a[i] < p);

            // move right index
            do {
                j--;
            } while (j >= l && a[j] > p);

            // swap elements if possible
            if (i < j)
                swap(a, i, j);
        } while (i < j);

        // at end of iteration, move pivot into correct position
        swap (a, i, r);
        // do quicksort for lower and greater subarrays
        quickSort(a, l, i - 1);
        quickSort(a, i + 1, r);
    }
}
```

- *example using rightmost element as pivot*:
    - *current array*: [10, 5, 19, 1, 14, 3]
    - *pivot*: 3
        - *swapped 10 (first greater than 3 from left) and 1 (first smaller than 3 from right)*: [1, 5, 19, 10, 14, 3]
    - *new array*: [1][3][19, 10, 14, 5]
    - *pivot*: 5
    - *new array*: [1][3][5][10, 14, 19]
    - *pivot*: 19
    - *new array*: [1][3][5][10, 14][19]
    - *pivot*: 14
    - *final*: [1][3][5][10][14][19]

RadixSort

- **runtime always $O(k \cdot n)$ with number of keys n and key length k, space complexity $O(n + k)$**
- **idea**: create and distribute elements into buckets according to their radix, then merge buckets and continue with new radix

    - *for decimal numbers*: from rightmost digit to leftmost digit, create buckets for each present digit (0-9), distribute numbers into corresponding buckets, merge buckets and repeat for next digit of number
    - *for words*: from rightmost letter to leftmost letter, create buckets for each present letter (A-Z), distribute words into corresponding buckets, merge buckets and repeat for next letter of word
- *example*:
    - *array*: 01**2**, 20**3**, 00**3**, 07**4**, 02**4**, 01**7**, 11**2**
    - *buckets (rightmost digit)*: {01**2**, 11**2**}, {20**3**, 00**3**}, {07**4**, 02**4**}, {01**7**}
    - *array*: 0**1**2, 1**1**2, 2**0**3, 0**0**3, 0**7**4, 0**2**4, 0**1**7
    - *buckets (middle digit)*: {2**0**3, 0**0**3}, {0**1**2, 1**1**2, 0**1**7}, {0**2**4}, {0**7**4}
    - *array*: **2**03, **0**03, **0**12, **1**12, **0**17, **0**24, **0**74
    - *buckets (leftmost digit)*: {**0**03, **0**12, **0**17, **0**24, **0**74}, {**1**12}, **{2**03}
    - *final*: 003, 012, 017, 024, 074, 112, 203

HeapSort

- **in-place, unstable, runtime always** $O(n \log n)$**, space complexity** $O(1)$
- uses **min-heap**, sorts in **reverse order** (*lowest to highest*)

```
HeapSort(Object[] H, int n) {
    // build min-heap from array
    build(H[0], ... , H[n - 1]);
    // deleteMin until heap empty
    for (i = n - 1; i >= 1; i--) {
        swap(H, 0, i);
        H.length--;
        siftDown(H, 0);
    }
}
```

- *example*[1]

Summary of Sorting Algorithm Complexities

Algorithm	Best Case	Average Case	Worst Case	Space Complexity
SelectionSort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
InsertionSort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$
MergeSort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(n)$
QuickSort	$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$O(n)$
RadixSort	$O(nk)$	$O(nk)$	$O(nk)$	$O(n+k)$
HeapSort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(1)$

Selection using QuickSelect

- *idea*: find k-th smallest element in array of n elements (*numbering starts at 1*)
  - similar to QuickSort, but we only look at one partition of the array
  - if k is smaller than the index of the pivot element (*also starting at 1*), continue with left array and same k
  - if k is greater than the index of the pivot element, continue with right array and $k = k - |a| - |b|$, where $|a|$ is the length of the left partition and $|b|$ is the length of the middle partition (*containing elements equal to pivot*)
  - else, element found
- *example - finding 7th smallest element in array (5)*
  - s = [3,1,4,1,5,9,**2**,6,5,3,5,8,9], k = 7 $\longrightarrow$ [1,1][2]**[3,4,5,9,6,5,3,5,8,9]**
  - s = [3,4,5,9,**6**,5,3,5,8,9], k = 4 $\longrightarrow$ **[3,4,5,5,3,5]**[6][9,8,9]
  - s = [3,4,**5**,5,3,5], k = 4 $\longrightarrow$ [3,4,3]**[5,5,5]**[] $\longrightarrow$ *found: 5*

Recursion Analysis and Master Theorem

- **divide-and-conquer algorithms**: algorithms that recursively *divide* the problem into smaller subproblems, that are then solved (*conquered*) and *merged* back together
- runtime analysis of recursive functions is done using **recurrence relations**
  - recurrence relations define **one or more base cases** and a **function to determine the rest**
  - e.g. Fibonacci numbers $F(x) = \begin{cases} 1 & \text{if n} = 0 \\ 1 & \text{if n} = 1 \\ F(n-2) + F(n-1) & \text{if n} > 1 \end{cases}$
- to solve recurrence relation, we need to get rid of the function's recursiveness and find a **closed form**
  - e.g. closed form of $F(x) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$
- **method 1**: iterative insertion
  - write first few steps by hand and try to deduce closed formula
  - e.g. $T(n) = \begin{cases} a & \text{if n} = 0 \\ T(n-1) + n & \text{if n} > 0 \end{cases}$
    - $T(1) = T(0) + 1 = a + 1$
    - $T(2) = T(1) + 2 = a + 1 + 2$
    - $T(3) = T(2) + 3 = a + 1 + 2 + 3$
    - *generalized*: $T(n) = a + (1 + 2 + ... + n) = a + \frac{n(n+1)}{2} \in O(n^2)$
- **method 2**: guess $f(n)$ then prove by induction
  - just wing it™
  - e.g. $T(n) = \begin{cases} 3 & \text{if n} = 1 \\ T(n-1) + 2^n & \text{if n} > 1 \end{cases}$
    - intuitively, guess that $f(n) = 2^{n+1} - 1$ and that $T(n) \leq f(n)$ for $n \geq 1$
    - in this case, we can prove that $T(n) = f(n)$
      - *induction basis*: for $n = 1$, $T(1) = 3 = 2^{1+1} - 1$
      - *induction hypothesis*: $T(n) = f(n)$ holds for some fixed $n \in \mathbb{N}$
      - *induction step*: prove that $T(n+1) = f(n+1)\ldots$
    - as such, $T(n) \in \Theta(f(n))$, i.e. $T(n) \in \Theta(2^n)$
- **method 3**: master theorem
  - follows a generalized formula of recurrence relations

- $T(n) = \begin{cases} a & \text{if n} = 1 \\ d \cdot T\left(\frac{n}{b}\right) + f(n) & \text{if n} > 1 \end{cases}$
  - $a \in \Theta(1)$: **runtime** for **base case** (*conquer*)
  - d: number of **new subproblems** per recursive layer
  - b: **factor**, by which the size of new subproblems per recursive layer is **reduced**
  - $f(n) = c \cdot n \in \Theta(n)$: runtime needed by current layer for **dividing** and **merging**
- $T(n) = \begin{cases} \Theta(n) & \text{if d} < \text{b} \\ \Theta(n \log n) & \text{if d} = \text{b} \\ \Theta(n^{\log_b d}) & \text{if d} > \text{b} \end{cases}$
- *example (mergesort)*:
  - MergeSort splits the array in 2 (d) of size $\frac{n}{2}$ $\left(\frac{n}{b}\right)$ each for each recursive layer $\rightarrow d = 2, b = 2$
  - the runtime of the base case is constant $\rightarrow a \in \Theta(1)$
  - the runtime of dividing and merging for each layer is linear $\rightarrow f(n) \in \Theta(n)$
  - $T(n) = \begin{cases} a & \text{if n} = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + f(n) & \text{if n} > 1 \end{cases}$
  - **since $d = b$, then $T(n) \in \Theta(n \log n)$**

Data Structures

Priority Queues

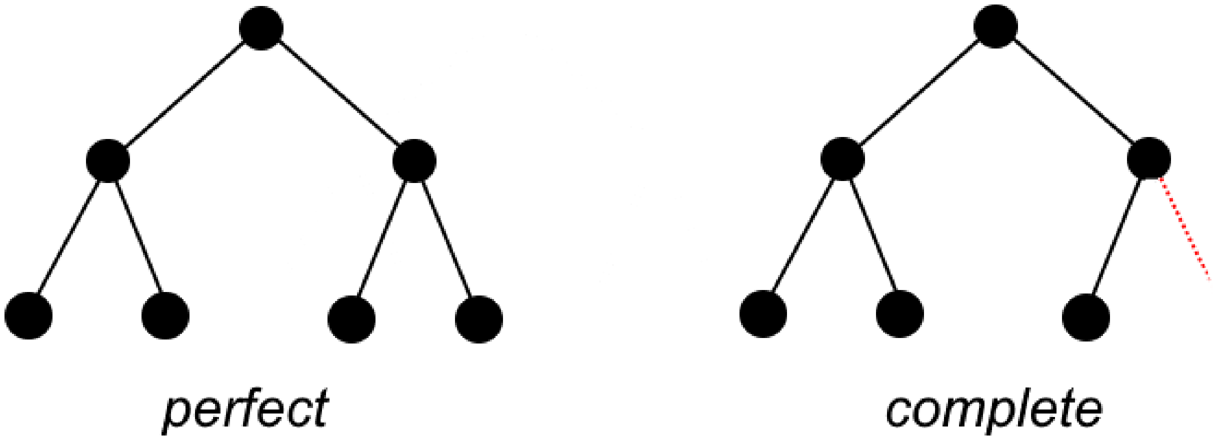
- abstract datatype, where each element is given a *priority*

Operation	unsorted list	sorted list
`build()`	$O(n)$	$O(n \log n)$
`insert()`	$O(1)$	$O(n)$
`min()`	$O(n)$	$O(1)$
`deleteMin()`	$O(n)$	$O(1)$

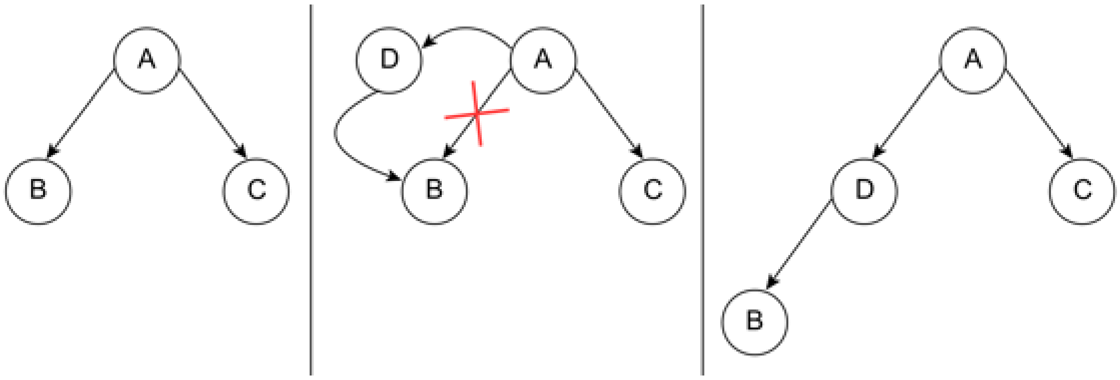
Binary Tree

- **tree data structure**, where each node (at most) has a **left and a right child** (*which themselves are binary trees*)
  - **leaf**: node without children
  - **inner node**: node with at least one child
  - **depth** t: number of edges from root to node (*root depth 0*)

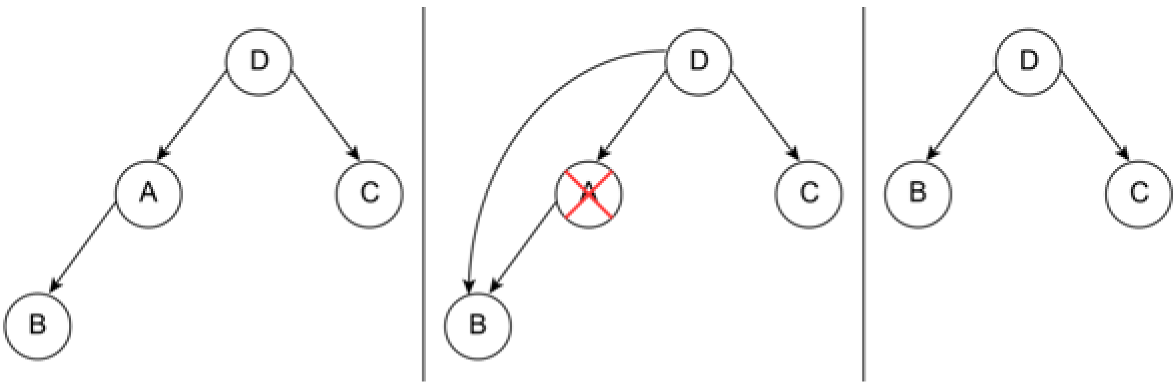
    - **height** h: depth from lowest node to root plus one (*starting height 1*)
    - **perfect binary tree**: $2^h - 1$ nodes, 2^{h-1} leaves
        - a full binary tree with n nodes has height $\lfloor \log_2(n) \rfloor + 1$
    - **complete binary tree**: the first $t - 1$ levels make up a complete binary tree, there exists a node e on level t such that there are no more nodes to the right of it



- **modifying a binary tree**:
    - `insert()`: $O(\log n)$



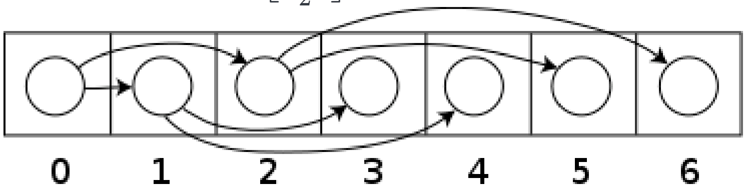
    - `delete()`: $O(\log n)$



Binary Heaps

- binary tree with…
    - **form invariant**: all layers are **complete** except for lowest layer
    - **heap invariant (min-heap)**: $\text{key}(\text{v.parent}) \leq \text{key}(\text{v})$

- can be stored using arrays, where a node with index i has children at indices $2i+1$ and $2i+2$ and parent node at $\lfloor \frac{i-1}{2} \rfloor$



- `deleteMin()`: replace root with last element in heap and sift down until heap invariant fulfilled, $\mathcal{O}(1)$ + runtime of `siftDown(v)`

```
// pseudocode
Element deleteMin(Heap<Element> H) {
    Element min = root of H;
    replace root of H with last element of H;
    siftDown(H, root of H);
    return min;
}
```

- `siftDown(v)`: move node down according to min-heap invariant, $\mathcal{O}(\log n)$

```
// pseudocode
siftDown(Heap<Element> H, Node v) {
    // cannot sift down if node is leaf
    if (isLeaf(v)) return;

    Node m;
    // choose direction
    if (key(v.left) < key(v.right)){
        m = v.left;
    }
    else {
        m = v.right;
    }

    // restore heap invariant or quit
    if (key(m) < key(v)) {
        swap content of m and v;
        siftDown(H, m);
    }
}
```

- `insert(e)`: insert element at end of heap then sift up into place, $\mathcal{O}(1)$ + runtime of `siftUp()`

```
// pseudocode
insert(Heap<Element> H, Element e) {
```

```
    Node v = insert e at end of H;
    siftUp(H, v);
}
```

- `siftUp(v)`: move node up according to min-heap invariant, $\mathcal{O}(\log n)$

```
// pseudocode
siftUp(Heap<Element> H, Node v) {
    while (v is not root && key(v.parent) > key(v)) {
        swap content of v and v.parent;
        v = v.parent;
    }
}
```

- `build(e1...en)`:
  - insert n elements unsorted into heap
  - do `siftDown()` for each node v on layer t bottom-up in reverse order (*right to left*)
    - *in other words*: the first $\lfloor n/2 \rfloor$ elements of the actual array, handled in reverse order (*e.g. for `[15,20,9,1,11,8,4,13,17]`, one would do `siftDown()` for `1,9,20,15` in that order*)
- `decreaseKey(v,k)`: $\mathcal{O}(\log n)$

```
decreaseKey(Heap<Element> H, Node v, int k) {
    if (k > key(v)) error();
    key(v) = k;
    siftUp(H, v);
}
```

- `increaseKey(v,k)`: $\mathcal{O}(\log n)$

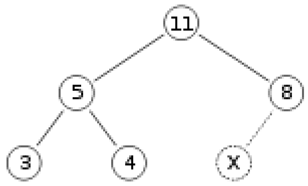
```
increaseKey(Heap<Element> H, Node v, int k) {
    if (k < key(v)) error();
    key(v) = k;
    siftDown(H, v);
}
```

- `delete(v)`: replace v with last node v' in heap then do `siftUp(v')` or `siftDown(v')`

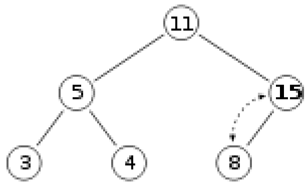
Binary Heap example using Max-Heap

- *insertion*: adding 15 into heap by inserting it at the end, then sifting up until heap invariant (here max-heap, i.e. $\text{key}(\text{parent}) > \text{key}(\text{child})$) is restored

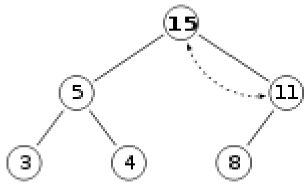
- *for visualization*: let X be the spot where 15 will be inserted at first



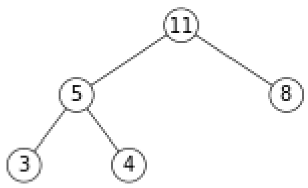
- place 15 there and check, if heap invariant is maintained $\longrightarrow$ since heap invariant is violated, sift 15 up and check again



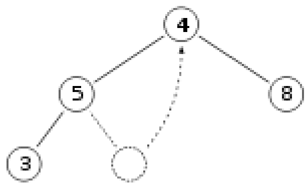
- since the heap invariant is violated once more, sift up once again $\longrightarrow$ since the node is now at the root, we have successfully inserted the node into the heap



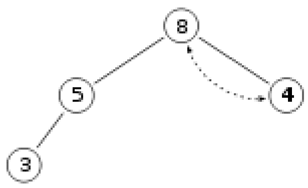
- *deletion*: using the same max-heap as before
  - let 11 be the node we want to remove (equiv. `deleteMax()`)



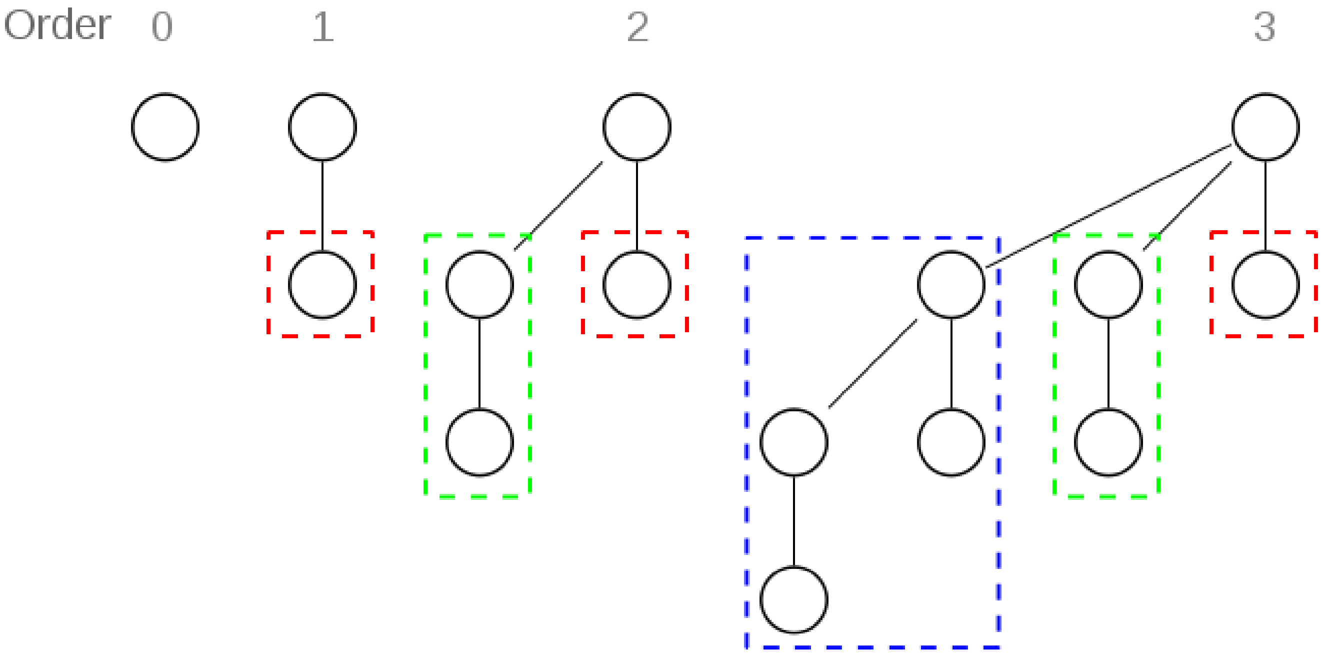
  - replace 11 with last node in heap, 4



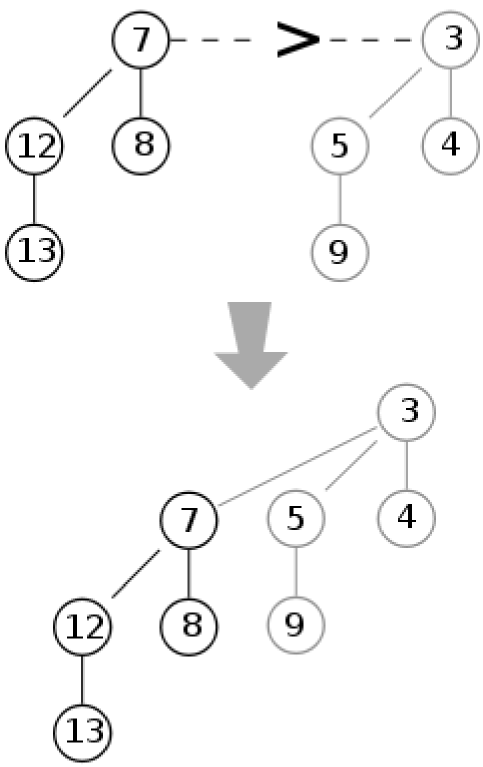
  - sift down, then heap invariant is restored



Binomial Trees



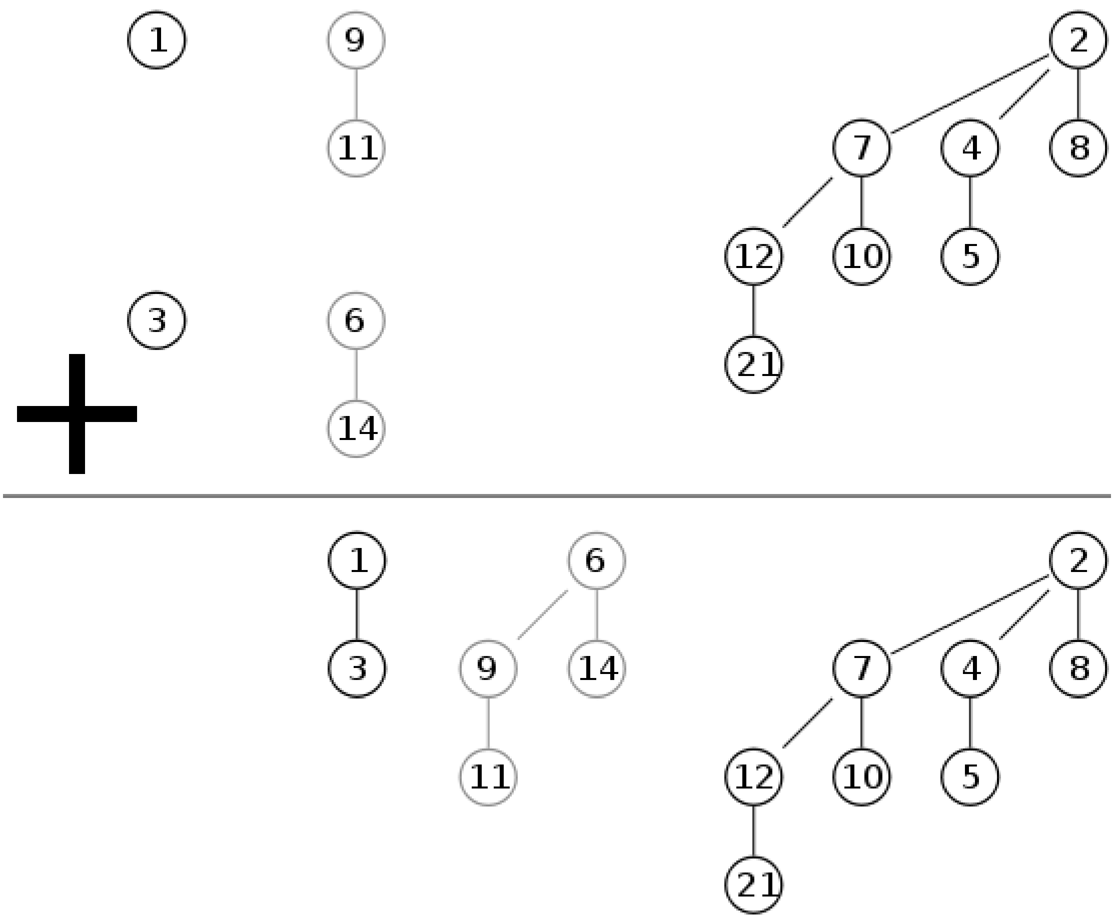
- a binomial tree of rank r has a root node with children of rank $r-1$, $r-2$, … , 0 in that order[2]
  - maximum depth r
  - depth $l \in \{0, \ldots, r\}$ has $\binom{r}{l}$ nodes ($\frac{r!}{l!(r-l)!}$)
  - in total 2^r nodes
  - maximum degree r in root
- **merging**: root node with bigger key becomes new child of root with smaller key[3]



- **removing root:** new binomial trees of ranks $r-1$ down to 0 appear

Binomial Heaps

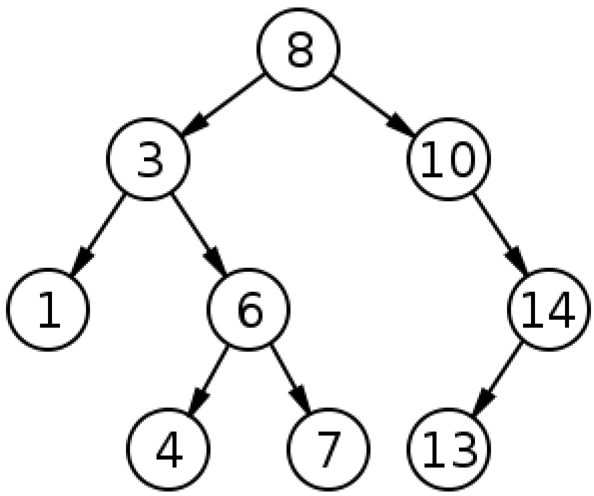
- set of binomial trees where **each tree fulfills the min-heap invariant**, there are **no two trees with the same rank** and a **min-pointer** points to the root with the smallest key
- a binomial heap with n nodes contains at most $1 + \lfloor \log_2(n) \rfloor$ binomial trees
  - the binary representation of n tells us exactly the rank of the trees in our heap
    - z.B. $n = 11_{10} = 1011_b = 1 * 2^3 + 1 * 2^1 + 1 * 2^0 \rightarrow$ there are trees of ranks 3,1,0
- **merging**: equivalent to binary addition, $\mathcal{O}(\log n)$ with $n = \max\{n_1, n_2\}$[4]



- **operations**:
  - `min()`: return root with minimal key (*located at min-pointer*)
    - $\mathcal{O}(1)$
  - `merge()`: equivalent to binary addition
    - $\mathcal{O}(\log n)$
  - `insert(e)`: `merge()` with binomial tree of rank 0, containing only `e`
    - $\mathcal{O}(\log n)$
  - `deleteMin()`: remove min-root and `merge()` the children with the rest of the heap
    - $\mathcal{O}(\log n)$

- `decreaseKey(v,k)`: set `key(v) = k`, then `siftUp()` in binomial tree of `v` and adjust min-pointer if needed
  - $\mathcal{O}(\log n)$
- `remove(v)`: first `decreaseKey(v, -inf)`, then `deleteMin()`
  - $\mathcal{O}(\log n)$

Binary Search Tree



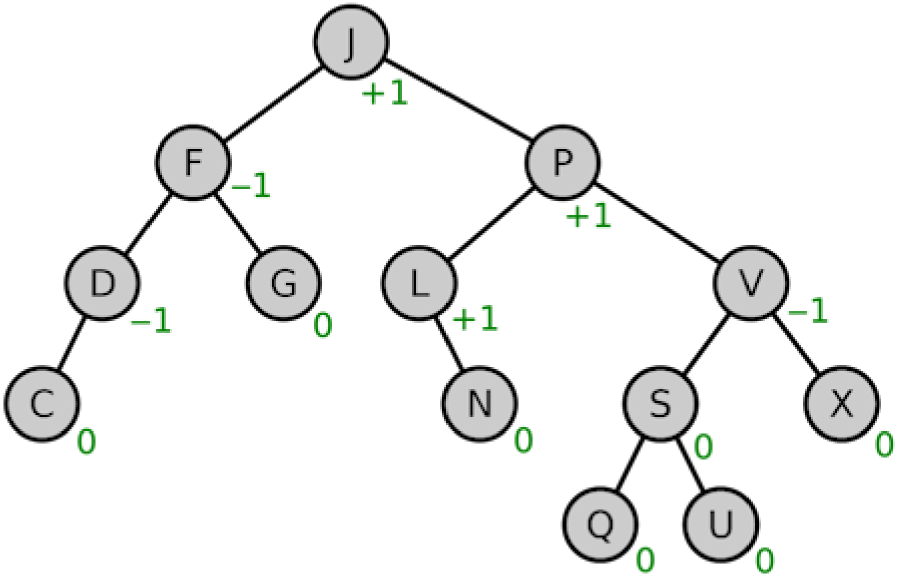
- **binary tree** with…
  - **search tree invariant**: left child smaller than parent, right child larger than parent
  - **key invariant**: each key is unique
  - **degree invariant**: a node can only have at most 2 children
- `locate(e)`: begin at root w of tree, $O(\log n)$
  - if $\text{key}(v) \geq k$, go to left child, else go to right child
  - return minimal node for which its key is greater or equal e
- `insert(e)`: $O(\log n)$
  - do `locate(key(e))` until e' is reached
  - if $\text{key}(e') > \text{key}(e)$, insert e before e' in list, and create new search tree node with $\text{key}(e)$ as splitter key to fulfill search tree invariant
  - else, throw error
- `remove(k)`: $O(\log n)$
  - do `locate(k)` until e is reached
  - if $\text{key}(e) = k$
    - delete e from list
    - delete parent key v from tree
    - if not already deleted, replace node with next smaller node in tree

    - else, throw error

- *cba with making original graphics here, just look in the slides or google it, the examples are good enough*

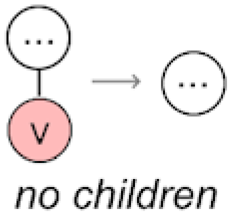
AVL-Trees

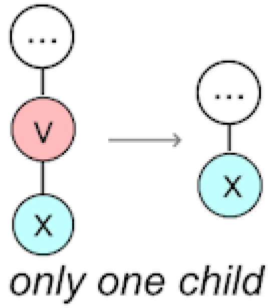
- **self-balancing binary search trees**
    - fixes and height-balancing possibly needed after insertion and deletion
- for any node, the **height** of its two subtrees differs by **at most 1**
    - **balance factor** = height of right subtree - height of left subtree $\in \{-1, 0, 1\}$[5]

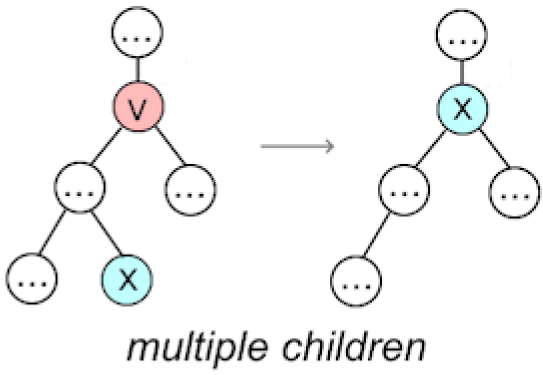


- **time complexity**: worst-case $O(\log n)$, best-case $\Theta(\log n)$
- **inserting**: start at root; if $k_{current} \geq v$, go left, otherwise right; insert where free space available then *rotate*
    - **left rotation** if balance factor of node is 2 and balance factor of right child is $+1$ or 0
    - **right rotation** if balance factor of node is -2 and balance factor of left child is -1 or 0
    - **right-left rotation** if balance factor of node is $+2$ and right child has balance factor -1
    - **left-right rotation** if balance factor of node is -2 and left child has balance factor of $+1$
- **deleting**:
    - if node **does not have a left** child, move **right** child into its place
    - if node **does not have a right** child, move **left** child into its place

- if node **has left and right** child, **replace with node with next smaller key**







- **balancing afterwards**: same as before

- **searching** works the same as in any standard binary search tree

Summary: AVL-Tree Rotations

- if **height differences** for parent and child have the **same sign**, perform **single rotation**
  - if **positive**, perform **left rotation**
  - if **negative**, perform **right rotation**
- if **height differences** for parent and child have **different signs**, perform **double rotation**
  - if **+2 / -1**, perform **R-L–rotation**
  - if **-2 / +1**, perform **L-R-rotation**

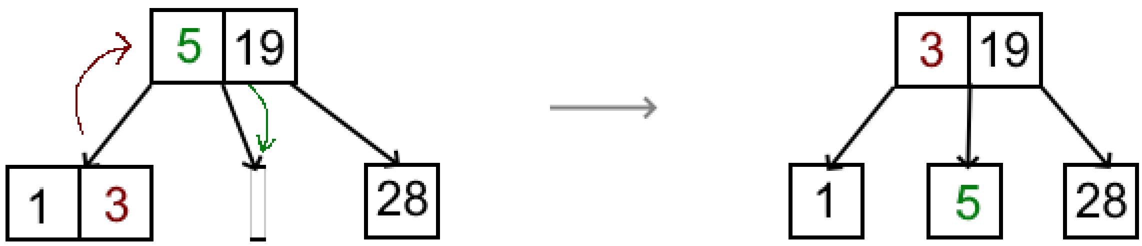
```
Source: https://www.geeksforgeeks.org/insertion-in-an-avl-tree/
T1, T2, T3 and T4 are subtrees.
S - single, D - double

S: RIGHT ROTATE
         z                                      y
        / \                                   /   \
       y   T4      Right Rotate (z)          x      z
      / \          - - - - - - - - ->      /  \    /  \
     x   T3                               T1   T2  T3   T4
    / \
  T1   T2

```

```
D: LEFT-RIGHT ROTATE
     z                               z                           x
    / \                            /   \                        /  \
   y   T4  Left Rotate (y)        x    T4  Right Rotate(z)    y      z
  / \      - - - - - - - - ->    /  \      - - - - - - - ->  / \    / \
T1   x                          y    T3                    T1  T2 T3  T4
    / \                        / \
  T2   T3                    T1   T2

S: LEFT ROTATE
  z                                y
 /  \                            /   \
T1   y     Left Rotate(z)       z      x
    /  \   - - - - - - - ->    / \    / \
   T2   x                     T1  T2 T3  T4
       / \
     T3  T4

D: RIGHT-LEFT ROTATE
   z                            z                            x
  / \                          / \                          /  \
T1   y   Right Rotate (y)    T1   x      Left Rotate(z)   z      y
    / \  - - - - - - - - ->     /  \   - - - - - - - ->  / \    / \
   x   T4                      T2   y                  T1  T2  T3  T4
  / \                              /  \
T2   T3                          T3   T4
```

(a,b)-Trees

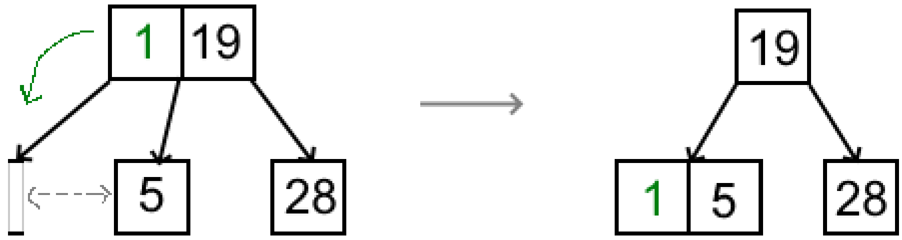
- variable definitions:
  - w: root
  - n leaves
  - $d(v)$: number of children (ext. degree) of a node v
  - $t(v)$: depth of a node v
- a **search tree** G is called an (a,b)-tree for $a \geq 2$ and $b \geq 2a - 1$ (alt. $a \leq \frac{b+1}{2}$) if following invariants are fulfilled:
  - **form invariant**: all leaves are at the **same depth**
  - **degree invariant**: for all internal nodes except for the root, $a \leq d(v) \leq b$
    - *in other words*, each node (except for the root) has at least a and at most b children
    - *for the root node*: $2 \leq d(w) \leq b$ (*except if it's a leaf*)

- depth $d \leq 1 + \lfloor \log_a \frac{n+1}{2} \rfloor$ if $n > 1$
- **all operations** $\Theta(\log n)$
- `locate(k)` works the same as in any search tree
- `insert(e)`:
  - locate e' using `locate(key(e))`
  - if $\text{key}(e) < \text{key}(e')$, insert e before e', otherwise throw error
  - insert $\text{key}(e)$ and handle in v into tree
    - **case 1**: if $d(v) \leq b$, **finish**
    - **case 2**: if $d(v) > b$, **split** v in two and **move splitter key** (*usually key at index* $\lfloor b/2 \rfloor$ *or median*) up
    - **case 2.5**: if degree of parent node is now bigger, **continue** until eventually $\deg \leq b$ or root has been split
- `remove(e)`:
  - let v be the **node** of e
    - **case 1**: v contains e (*lowest depth*)
      - directly **delete** e and v
      - if v now has less than a children, **steal** or **merge**
    - **case 2**: v does not contain e (*not on lowest depth*)
      - let e' be the element directly before e, included in v
      - delete e' from v and e from list
      - replace remaining e in tree with e' (*i.e. replace key with value which contained pointer to* e)
      - if v now has less than a children, **steal** or **merge**
  - **steal** if neighbouring node v' of v **has more** than a children, start with left neighbour
    - v' left of v: rightmost key in v' goes **up**, replaced key goes **down** into v
    - v' right of v: leftmost key in v' goes **up**, replaced key goes **down** into v



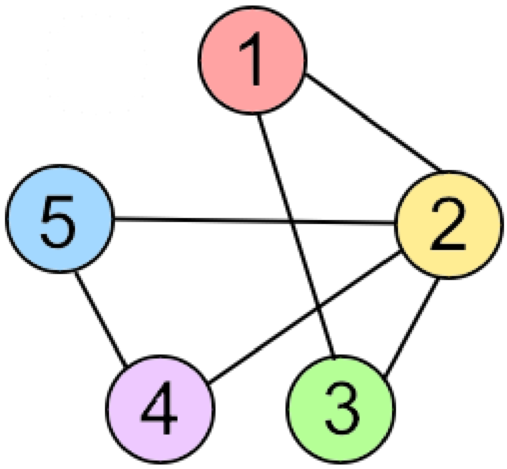
  - **merge** if neighbouring node v' of v **does not have more** than a children
    - merge v with neighbouring node, preferably left node, by bringing **down** father element

- father node and adjacent nodes may need to be adjusted with steal / merge too afterwards, since we're taking a node away from it
- if root is empty, remove it



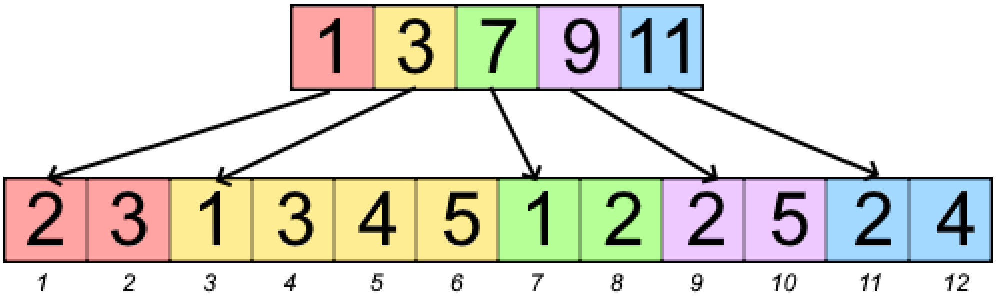
- *for the same reason as before, if you want examples, look in the slides and go along with those*

Graphs

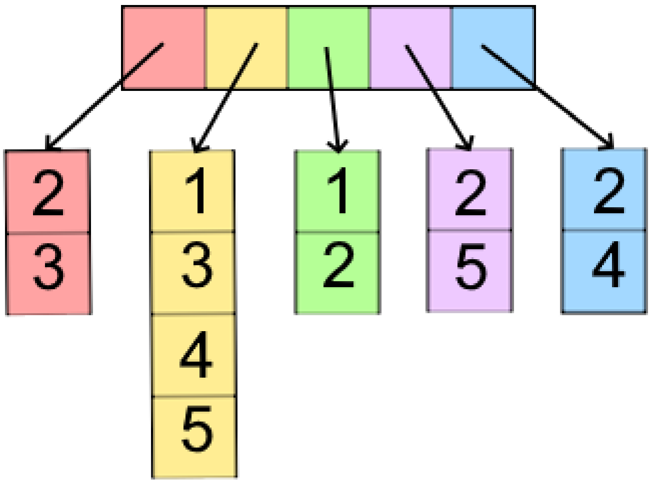


- **representing a graph**:
  - **list of edges**
    - $\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{2,5\},\{4,5\}$
    - **+**: $O(m)$ space complexity, `insert(Edge e)`, `insert(Node v)` and `remove(Key i)` in $O(1)$
    - **-**: `find(Key i, Key j)` and `remove(Key i, Key j)` worst-case $\Theta(m)$
  - **adjacency matrix**
    - $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$
    - **+**: can tell in $O(1)$ if two nodes are neighbors, inserting and deleting edges in $O(1)$
    - **-**: space complexity $\Theta(n^2)$, finding all neighbors of a node costs $O(n)$ time

- **adjacency arrays** (top: *indices* of node in bottom array, bottom: neighboring node *keys*)



  - **+**: space complexity $n + m + \Theta(1)$ for directed graphs and $n + 2m + \Theta(1)$ for undirected graphs
  - **-**: inserting and deleting edges is costly
- **adjacency lists** (*similar to arrays, but with linked lists*)

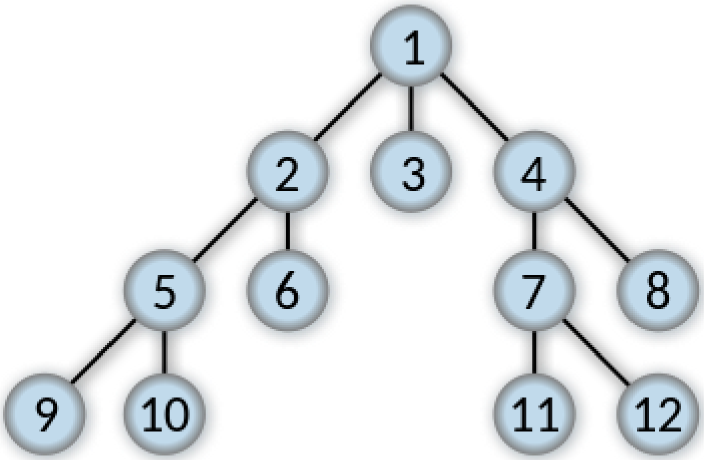


  - **+**: inserting edges in $O(1)$, deleting edges in $O(d)$ or $O(1)$ with handle
    - when using adjacency lists with a hash table, all operations can be done in $O(1)$
  - **-**: usage of lists requires heap space and generally takes longer
    - when using adjacency lists with a hash table, the space complexity becomes $O(n + m)$

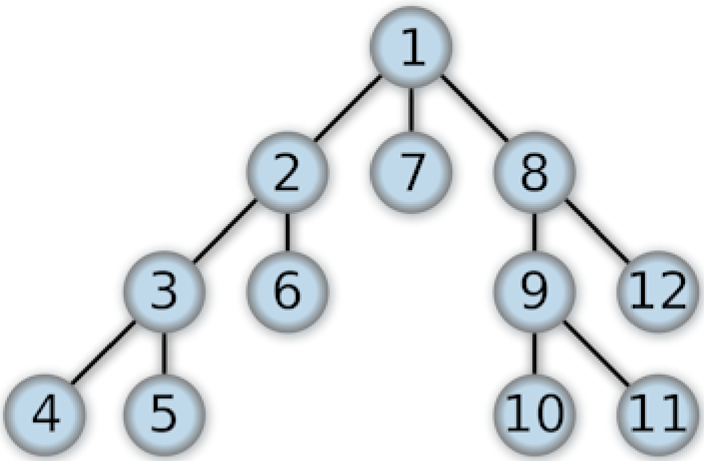
- **traversing a graph** ($O(|V| + |E|)$):
  - [breadth-first-search]
    - **horizontal** before vertical
    - operates based on a **FIFO-queue**
    - useful for SSSP (*single source shortest path*) due to storing distance of each node to source
    - **algorithm**:
      - insert node in queue
      - take front item of queue and add it to visited list
      - create list of vertex's adjacent nodes, add ones not yet visited to the back of the queue

- repeat steps 2 and 3 until queue is empty

- *order of expansion*[6]:

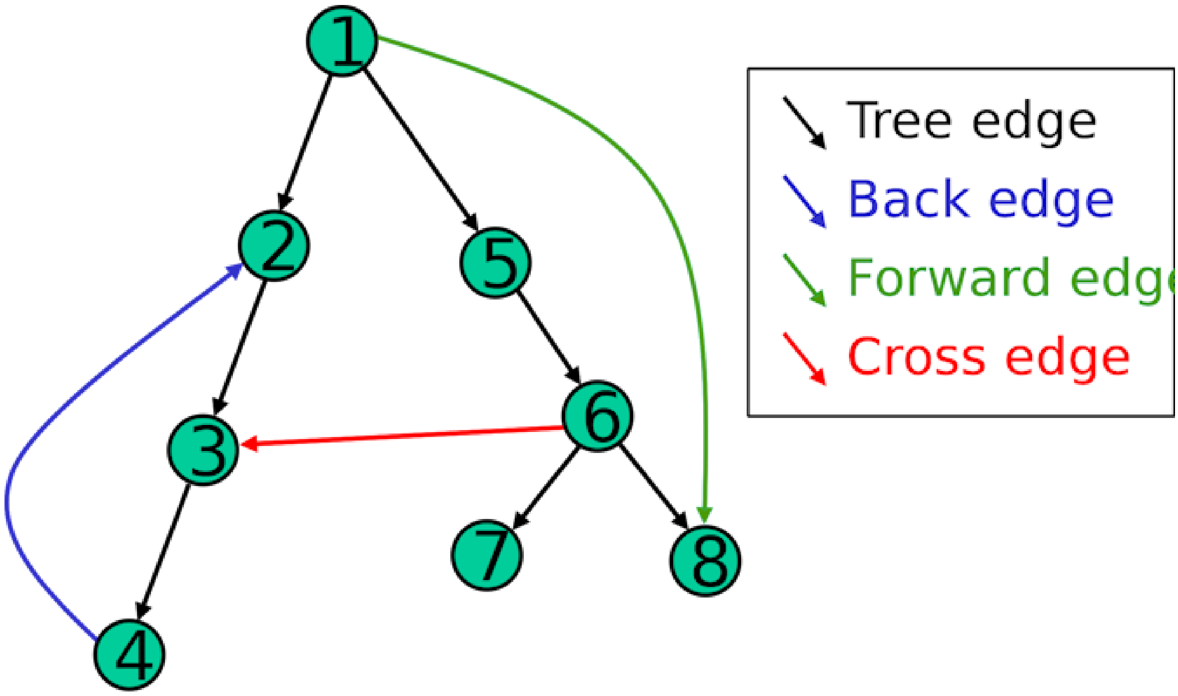


- [depth-first search]
  - **vertical** before horizontal
  - operates based on a **stack**
  - **algorithm**:
    - insert node onto stack
    - take top item of stack and add it to visited list
    - create list of vertex's adjacent nodes, add ones not yet visited to the top of the stack
    - repeat steps 2 and 3 until stack is empty
  - *order of expansion*[7]:



  - **extra variables**:
    - `dfsNum`: exploration order
    - `finishNum`: finished order
  - **types of edges**:
    - **root edge**: edge from root outwards

- **forwards edge**: to a successor
- **backwards edge**: to a predecessor



- **using DFS to recognize DAGs**:
  - DFS **does not contain any backwards edges**
  - for **all** edges (v, w), `finishNum[v] > finishNum[w]` (*higher finish number points to lower finish number*)

Type of Edge	`dfsNum[v] < dfsNum[w]`	`finishNum[v] > finishNum[w]`
Root Edge	yes	yes
Forwards Edge	yes	yes
Backwards Edge	no	no
Rest	no	yes

Connectivity

- a graph is **connected** if **every pair of vertices in the graph** is **connected** $\equiv$ there is a **path** between every pair of vertices
- a graph with just one vertex is **connected**
  - an edgeless graph with two or more vertices is disconnected
- a **directed graph** is called **weakly connected** if replacing all of its directed edges with undirected edges produces a connected (undirected) graph
- it is **unilaterally connected** if it contains a directed path from u to v **or** a directed path from v to u for every pair of vertices u, v
- it is **strongly connected**, or simply **strong**, if it contains a directed path from u to v **and** a directed path from v to u for every pair of vertices u, v

- a **connected component** is a **maximal connected subgraph of an undirected graph**
    - each vertex belongs to **exactly one** connected component, as does each edge
    - a graph is connected **if and only if** it has exactly one connected component
- the **strong components** are the **maximal strongly connected subgraphs of a directed graph**

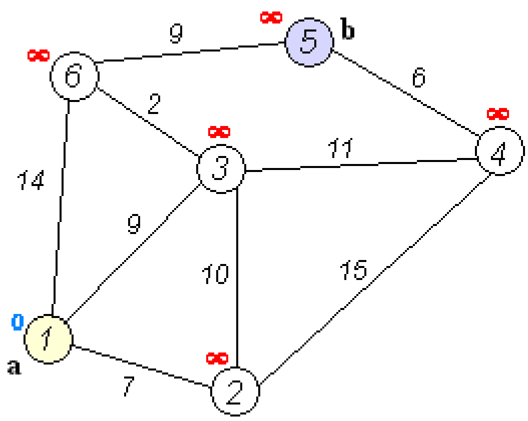
Shortest Paths (SSSP)

- **case 1**: edge costs 1 $\longrightarrow$ **BFS**
- **case 2**: DAG, variable edge costs $\longrightarrow$ **Topological Sorting**
- **case 3**: variable graph, positive edge costs $\longrightarrow$ **Dijkstra**
- **case 4**: variable graph, variable edge costs $\longrightarrow$ **Bellman-Ford**

DAG - Topological Sorting

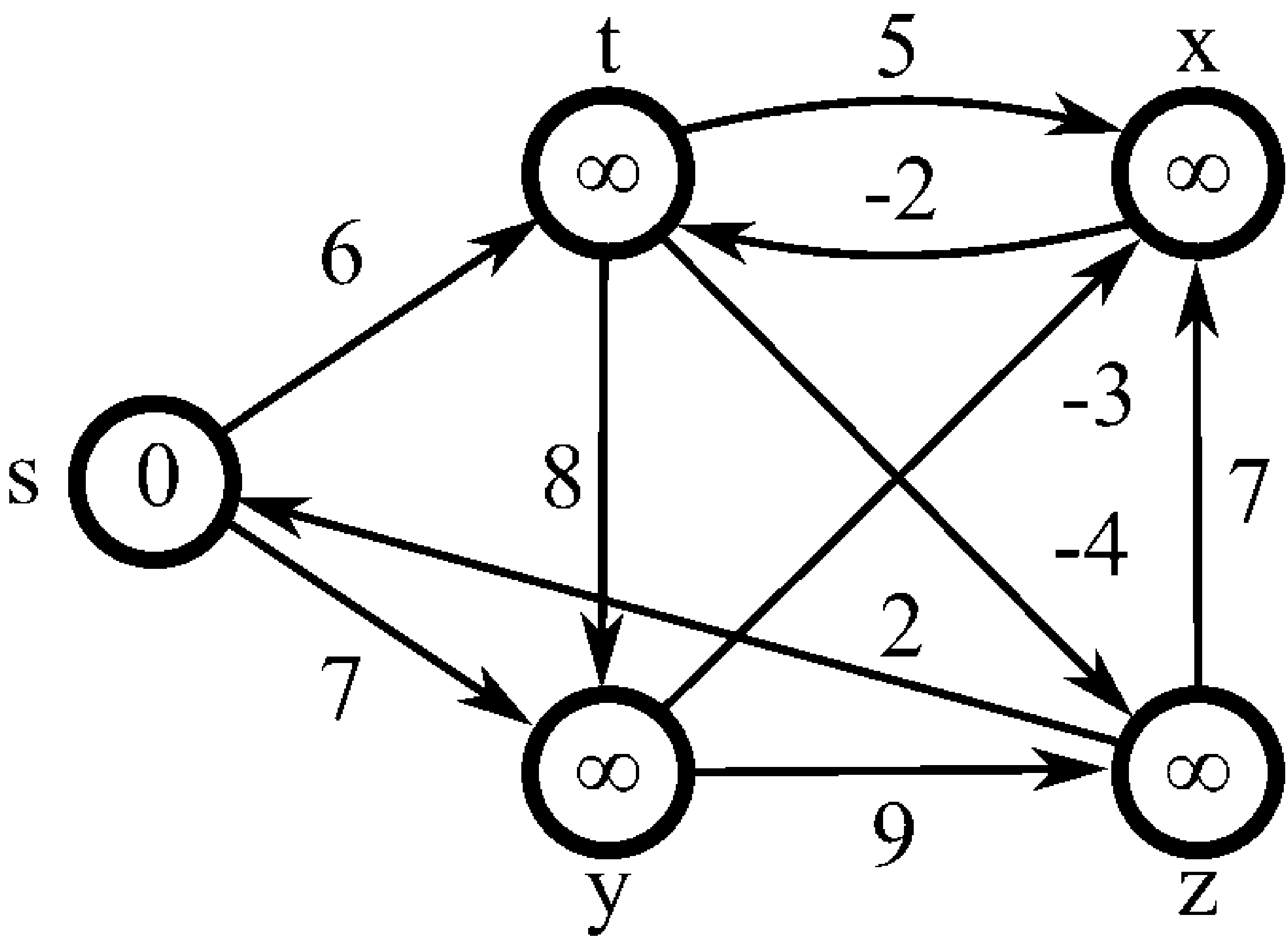
```
L ← Empty list that will contain the sorted elements
S ← FIFO-Queue of all nodes with no incoming edge

while S is not empty do
    remove a node n from S
    add n to L
    for each node m with an edge e from n to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S

if graph has edges then
    return error   (graph has at least one cycle)
else
    return L   (a topologically sorted order)
```

Dijkstra’s Algorithm



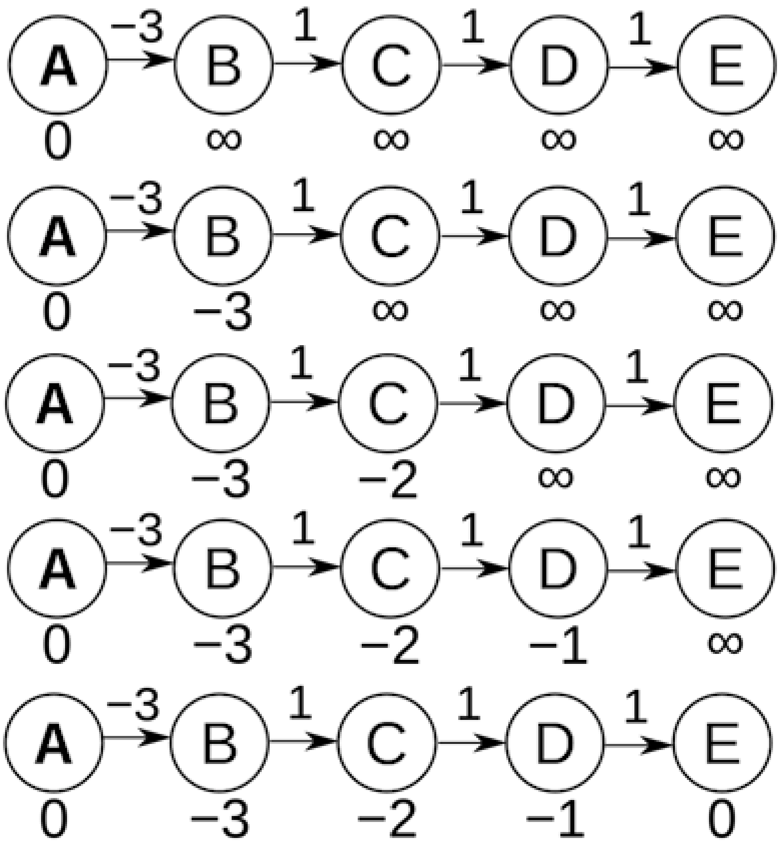
- used to find *shortest paths between nodes* in a weighted graph with *positive weights*
- **algorithm**:
  - mark all nodes unvisited and create set of unvisited nodes
  - assign tentative distances to each node (0 for initial node, ∞ for all other nodes)
    - the tentative distance of a node is the length of the shortest path (so far) between said node and the starting node
  - for the current node, calculate tentative distances of neighboring *unvisited* nodes through current node
    - if newly calculated tentative distance is smaller than current distance, replace current distance with tentative distance
  - mark current node as visited (*remove from unvisited set*)
    - if destination node is marked as visited or if smallest tentative distance among nodes in unvisited set is infinity, *stop*
    - else, go to unvisited node with smallest tentative distance and go back to third step
- **time complexity**: $O(|E| + |V| \log |V|)$

Bellman-Ford[8]



- works on graphs with **negative edge weights**

- **fundamental idea**: there are **at most** $|V|-1$ edges in one of our paths (*because if there were* $|V|$ *or more, there would be a cycle*)
- **algorithm**:
  - initialize distance to source to 0 and all other nodes to infinity
  - *for all edges*: if the distance to the destination can be shortened by taking the edge, the distance is updated to the new lower value
    - if $dist[v] > dist[u] + weight((u,v))$, then $dist[v] = dist[u] + weight((u,v))$
  - repeat last step $|V|-1$ times
    - if in the last iteration, distances are *still* being updated, then finally update these distances to $-\infty$, indicating that there is a *negative weight cycle*



- **time complexity**: $O(|E| \cdot |V|)$

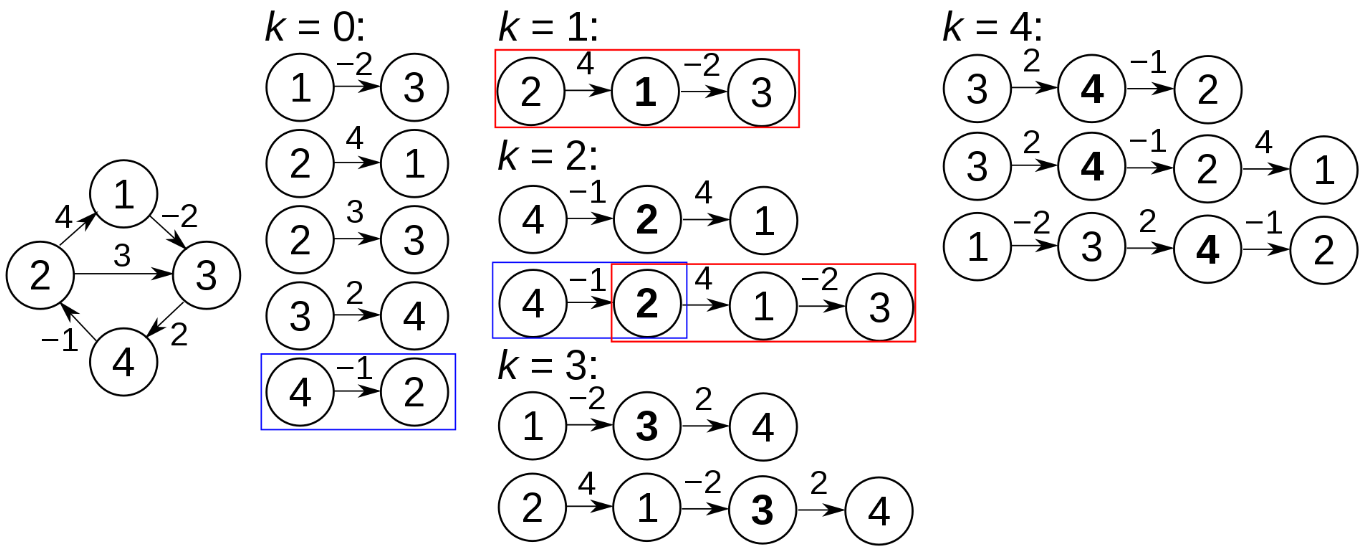
Shortest Paths (APSP)

Floyd-Warshall's Algorithm

- $O(n^3)$, please don't use this

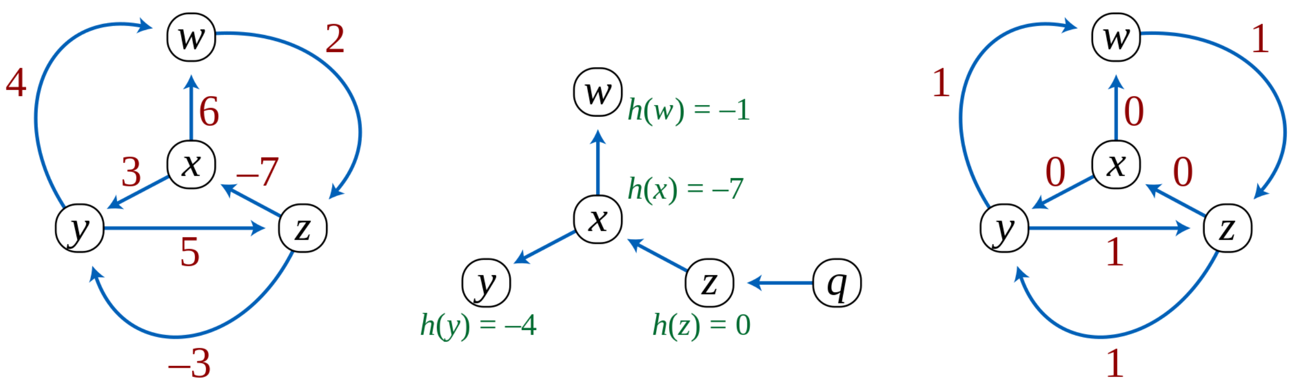
```
let dist be a |V| × |V| array of minimum distances initialized to ∞
(infinity)
for each edge (u, v) do
    dist[u][v] ← w(u, v)
for each vertex v do
```

```
    dist[v][v] ← 0
for k from 1 to |V|
    for i from 1 to |V|
        for j from 1 to |V|
            if dist[i][j] > dist[i][k] + dist[k][j]
                dist[i][j] ← dist[i][k] + dist[k][j]
            end if
```



Johnson's Algorithm

- insert new **temporary node** s with edge (s, v) to all v and $c(s, v) = 0$
- calculate $d[s, v]$ using **Bellman-Ford's Algorithm** and set $\phi[v] = d[s, v]$ for all v
- calculate **modified edge costs** $\overline{c}(e) = \phi(v) + c(e) - \phi(w)$
- calculate $\overline{d}[v, w]$ for all v without s using **Dijkstra's Algorithm** using the modified costs
- calculate **proper distances** $d[v, w] = \overline{d}[v, w] + \phi[w] - \phi[v]$
- *example: first 3 stages*



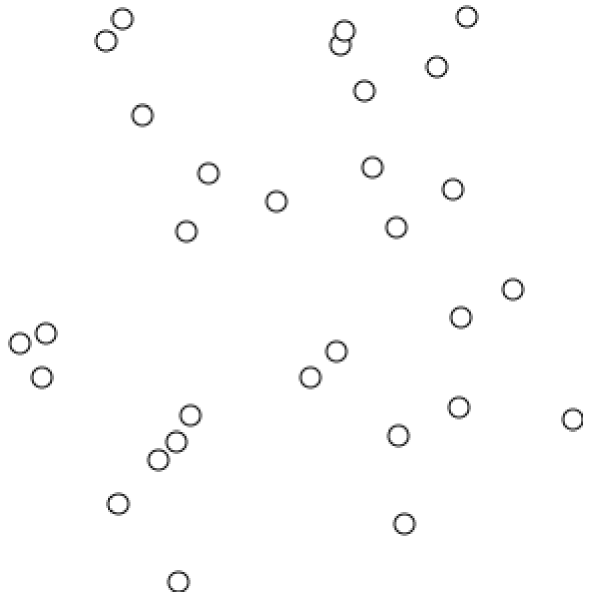
original graph with negative edges

shortest path tree found by Bellman-Ford

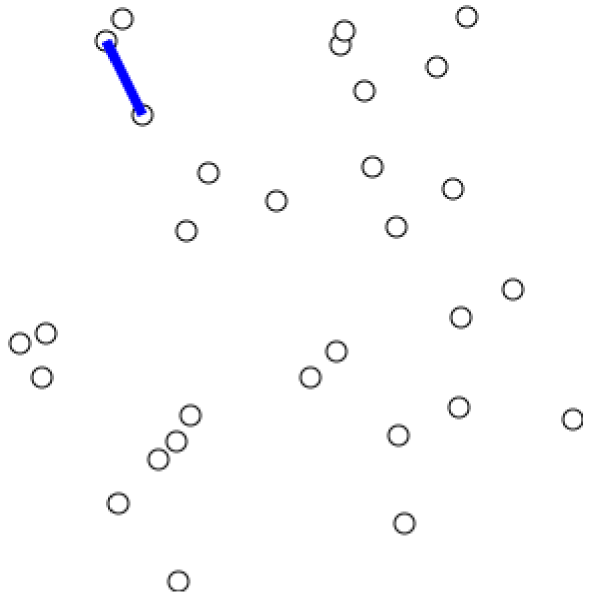
reweighted graph with no negative edges

Minimum Spanning Trees

- **Kruskal's Algorithm** ($O(m \log m)$):
  - repeatedly choose a minimum-cost edge connecting two connected components until **only one connected component** remains[9]



- **Prim's Algorithm**:
  - look at **growing tree** T, initially **consisting of any single node** s
  - add to T an edge **with minimal weight** from a **tree node** to a **node outside the tree** (*if there are multiple possibilities, it doesn't matter which*)
  - repeat selection until all n nodes in tree[10]



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